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CHARACTERIZATIONS OF LAGUERRE-HAHN AFFINE ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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ABSTRACT: In this work we characterize a monic polynomial sequence, orthogonal with respect to a hermitian linear functional u that satisfies a functional equation $\mathrm{D}(Au) = Bu + zH\mathcal{L}$, where A, B and H are polynomials and $\mathcal{L}$ is the Lebesgue functional, in terms of a first order linear differential equation for the Carathéodory function associated with u and in terms of a first order structure relation for the orthogonal polynomials.

KEYWORDS: Orthogonal polynomials on the unit circle, hermitian functionals, measures on the unit circle, semi-classical functionals, Carathéodory function.
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1. Introduction

Let u be a linear functional, defined in the linear space of polynomials with real coefficients. A linear functional u is Laguerre-Hahn affine if the corresponding formal Stieltjes function satisfies a first order linear differential equation,

$$\phi(x)S'(x) = B(x)S(x) + C(x) \tag{1}$$

with ϕ, B, C polynomials (*cf.* [11]). It is known that the Laguerre-Hahn affine class and the semi-classical class coincide. This result follows from [10, 12], where it is established that u is Laguerre-Hahn affine if, and only if, u satisfies a functional Pearson equation, $\mathrm{D}(\phi u) = \psi u$, where ϕ and ψ are polynomials (ϕ is the same polynomial as in (1)).

Laguerre-Hahn affine functionals on the real line are also characterized in terms of first order structure relations for the corresponding sequence of orthogonal polynomials on the real line, $\{P_n\}$,

$$\phi(x)P'_{n+1}(x) = C_n(x)P_{n+1}(x) + D_n(x)P_n(x),\ n = 0, 1, \ldots$$

where C_n, D_n are polynomials of bounded degree (*cf.* [10, 11, 12]).

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In [8, 18], an analogue theory for hermitian linear functionals defined in the linear space of Laurent polynomials with complex coefficients was outlined. The concept of semi-classical functional was extended to this set of functionals; a hermitian linear functional u is said to be semi-classical if it satisfies a Pearson equation $\mathrm{D}\,(Au) = Bu$, where A, B are polynomials (see, in section 3, the definition of the derivation operator D), and the corresponding sequences of orthogonal polynomials, semi-classical orthogonal polynomials on the unit circle, were defined; the Laguerre-Hahn affine class on the unit circle was defined in terms of a first order linear differential equation with polynomial coefficients for the formal series $G(z) = \sum_{n=-\infty}^{+\infty} c_n z^n$, where c_n is the n-th moment of the hermitian functional u,

$$A(z)G'(z) = B_1(z)G(z) + H(z). \tag{2}$$

Since then, the comparison between both theories, namely the characterization of the functionals in terms of differential properties for the corresponding sequences of orthogonal polynomials, as well the generating functions for the moments, has been the central theme in some works (see [1, 2, 9, 13, 16, 18]).

In [2, 13, 18], it is established that u is Laguerre-Hahn affine and the corresponding G satisfies (2) if, and only if, the corresponding u satisfies the generalized Pearson equation $\mathrm{D}\,(Au) = Bu + zH\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue operator and B is a polynomial depending on A, B_1. Moreover, in [2, 13], the authors obtain conditions on the coefficients of equation (2) and, also, on the polynomial coefficients of a differential equation for the formal Carathéodory function F,

$$A(z)F'(z) = B_1(z)F(z) + C(z) \tag{3}$$

in order to establish the semi-classical character of the corresponding functional. Some examples of functionals and the corresponding sequences of orthogonal polynomials that are not semi-classical are given, thus showing that, in the complex case, the Laguerre-Hahn affine class and the semi-classical class do not coincide.

In this work, following a different approach from the referred works, we study the relation between a first order differential equation for the Carathéodory function, (3), and the distributional equation for the corresponding u (we remind that, in many ways, the Carathéodory function is the analogue of the Stieltjes function (see [14])). We prove that if F satisfies a first order differential equation (3) in $|z| < 1$, then the corresponding linear functional u satisfies a generalized Pearson equation $\mathrm{D}\,(Au) = Bu + H\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue operator and B, H are polynomials given explicitly in terms of A, B_1, C. Then, we deduce first order structure relations for the corresponding sequences of orthogonal polynomials on the unit circle (analogue of the structure relations for orthogonal polynomials on the real line, studied in [10, 11, 12]). Finally, using these structure relations, we obtain a differential system for semi-classical orthogonal polynomials on the unit circle (the analogue of the result established for semi-classical orthogonal polynomials on the real line in [7], by Magnus).

This paper is organized as follows: in section 2 we give the definitions and state the main results which will be used in the next sections. In section 3 we study the relation between the first order linear differential equation for F, and the generalized Pearson equation for u (see theorem 3). In section 4, we establish the equivalence between a first order differential equation for F and a system of differential relations for the sequence of orthogonal polynomials, for the sequence of associated polynomials of the second kind and for the sequence of functions of the second kind. We deduce a differential system for sequences of semi-classical orthogonal polynomials on the unit circle.

2. Preliminary results

Let $\Lambda = \operatorname{span}\{z^k : k \in \mathbb{Z}\}$ be the space of Laurent polynomials with complex coefficients, Λ' its algebraic dual space, $\mathbb{P} = \operatorname{span}\{z^k : k \in \mathbb{N}\}$ the space of complex polynomials and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (or, using the parametrization $z = e^{i\theta}$, $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\}$) the unit circle.

Given a sequence of moments (c_n) of a linear functional $u : \Lambda \to \mathbb{C}$, $c_n = \langle u, \xi^{-n} \rangle$, $n \in \mathbb{Z}$, $c_0 = 1$, the minors of the Toeplitz matrix are defined by

$$\Delta_k = \begin{vmatrix} c_0 & c_1 & \cdots & c_k \\ c_{-1} & c_0 & \cdots & c_{k-1} \\ \vdots & \vdots & & \vdots \\ c_{-k} & c_{-k+1} & \cdots & c_0 \end{vmatrix}, \ \Delta_0 = c_0, \ \Delta_{-1} = 1, \ k \in \mathbb{N}$$

Definition 1. (*cf.* [17]). The linear functional u is:

a) *hermitian* if $c_{-n} = \overline{c_n}, \forall n \geq 0$,

b) *regular* or *quasi-definite* if $\Delta_n \neq 0, \forall n \geq 0$,

c) *positive definite* if $\Delta_n > 0, \forall n \geq 0$.

If u is a positive definite hermitian functional there exists a non-trivial probability measure μ supported on the unit circle such that

$$\langle u, \xi^{-n} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \xi^{-n} d\mu(\theta),\ n \in \mathbb{Z},\ \xi = e^{i\theta}.$$

Hereafter we will use the notation $\langle u_\theta,\ .\ \rangle$ to denote the action of the linear functional u over the variable θ, $\theta \in [0, 2\pi[$.

Definition 2. Let $\{\phi_n\}$ be a sequence of complex polynomials with $\deg(\phi_n) = n$ and u a hermitian linear functional. We say that $\{\phi_n\}$ is a *sequence of orthogonal polynomials* with respect to u (or $\{\phi_n\}$ is a sequence of orthogonal polynomials on the unit circle) if

$$\langle u, \phi_n(\xi)\overline{\phi_m}(1/\xi) \rangle = K_n \delta_{n,m},\ K_n \neq 0,\ n, m \in \mathbb{N},\ \xi = e^{i\theta}.$$

If the leading coefficient of each ϕ_n is 1, then $\{\phi_n\}$ is said to be a *sequence of monic orthogonal polynomials.*

It is well known (see [3, 4, 5]) that a given hermitian linear functional u is regular if, and only if, there exists a sequence $\{\phi_n\}$ of orthogonal polynomials with respect to u. Sequences of monic orthogonal polynomials $\{\phi_n\}$ satisfy each of the following recurrence relations, for $n \geq 1$,

$$\begin{aligned} (R_1)\ \ & \phi_n(z) = z\phi_{n-1}(z) + a_n \phi^*_{n-1}(z) \\ (R_2)\ \ & \phi^*_n(z) = \phi^*_{n-1}(z) + \overline{a}_n z \phi_{n-1}(z) \end{aligned}$$

with $a_n = \phi_n(0)$, and initial conditions $\phi_0(z) = 1,\ \phi_{-1}(z) = 0$, and the polynomials $\{\phi^*_n\}$ are defined by $\phi^*_n(z) = z^n \overline{\phi_n}(1/z),\ n = 0, 1, \ldots$, where $n = \deg(\phi_n)$. Also, $|a_n| \neq 1,\ \forall n \in \mathbb{N}$, in the regular case and $|a_n| < 1,\ \forall n \in \mathbb{N}$, in the positive-definite case.

We consider the formal series associated with the hermitian linear functional u (whose sequence of moments is (c_n) and $c_0 = 1$), and denote it by F,

$$F(z) = 1 + 2\sum_{k=1}^{+\infty} c_k z^k,\ |z| < 1\,, \quad F(z) = -1 - 2\sum_{k=1}^{+\infty} \overline{c}_k z^{-k}\,, \quad |z| > 1 \tag{4}$$

Since, for each $\theta \in [0, 2\pi[$, the following expansions take place,

$$\frac{e^{i\theta} + z}{e^{i\theta} - z} = 1 + 2\sum_{k=1}^{+\infty} (e^{i\theta})^{-k} z^k,\ |z| < 1\,,\ \ \frac{e^{i\theta} + z}{e^{i\theta} - z} = -1 - 2\sum_{k=1}^{+\infty} (e^{-i\theta})^k z^{-k},\ |z| > 1$$

then, formally,

$$\langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = F(z) \tag{5}$$

Thus, we will also say that the series in (4) correspond (formally) to the function F defined by (5). In the positive definite case, F is the Carathéodory function corresponding to u, and is represented by

$$F(z) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta),\ z \in \mathbb{C} \setminus \mathbb{T}$$

where μ is the probability measure associated with u.

Given a sequence of monic orthogonal polynomials $\{\phi_n\}$ with respect to u, the *sequence of associated polynomials of the second kind* $\{\Omega_n\}$ are defined by

$$\begin{aligned} &\Omega_n(z) = \langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\phi_n(e^{i\theta}) - \phi_n(z)\right)\rangle,\ n = 1, 2, \ldots \\ &\Omega_0(z) = 1. \end{aligned}$$

The associated polynomials $\{\Omega_n\}$ also satisfy recurrence relations,

$$\Omega_n(z) = z\Omega_{n-1}(z) - a_n\Omega^*_{n-1}(z),\ n = 1, 2, \ldots$$

with initial conditions $\Omega_0(z) = 1,\ \Omega_{-1}(z) = 0$.

The *functions of the second kind* associated with $\{\phi_n\}$ are defined by

$$\begin{aligned} &Q_n(z) = \langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\phi_n(e^{i\theta})\rangle,\ n = 1, 2, \ldots \\ &Q_0(z) = F(z) \end{aligned}$$

and $\{Q_n\}$ satisfy the following recurrence relations (*cf.* [15]),

$$Q_n(z) = zQ_{n-1} - a_nQ^*_{n-1}(z),\ n = 1, 2, \ldots$$

with $Q_0(z) = F(z)$ and $Q^*_0(z) = -F(z)$.

Theorem 1 (*cf.* [3, 4, 5])**.** *Let $\{\phi_n\}$ be a sequence of monic orthogonal polynomials on the unit circle and $\{\Omega_n\}$, $\{Q_n\}$ the sequence of the associated polynomials and the functions of the second kind, respectively. Then the* *following equations hold,* $\forall n \geq 1$,

$$Q_n(z) = \Omega_n(z) + F(z)\phi_n(z), \tag{6}$$
$$Q_n^*(z) = \Omega_n^*(z) - F(z)\phi_n^*(z) \tag{7}$$
$$\phi_n^*(z)\Omega_n(z) + \phi_n(z)\Omega_n^*(z) = 2h_n z^n \tag{8}$$
$$\phi_n^*(z)Q_n(z) + \phi_n(z)Q_n^*(z) = 2h_n z^n \tag{9}$$

with $h_n = \prod_{k=1}^{n}(1-|a_k|^2)$ *and* $Q_n^* = z^n\overline{Q_n}(1/z)$.

As a consequence we get the following results (see [15]).

Corollary 1. *Let* $\{Q_n\}$ *be the sequence of functions of the second kind associated with* $\{\phi_n\}$*. Then, the following holds,* $\forall n \geq 1$,

$$Q_n(z) = 2h_n z^n + \mathcal{O}(z^{n+1}),\ |z| < 1 \tag{10}$$
$$Q_n(z) = 2a_{n+1}h_n z^{-1} + \mathcal{O}(z^{-2}),\ |z| > 1\,. \tag{11}$$

Corollary 2. *Let* $\{\phi_n\}$ *be a sequence of monic orthogonal polynomials on the unit circle and* $\{\Omega_n\}$ *the sequence of associated polynomials of the second kind. Then, the following holds:*
a) If there exists $k \in \mathbb{N}$ *such that* $\phi_k(\alpha) = \Omega_k(\alpha) = 0$*, then* $\alpha = 0$*;*
b) If there exists $k \in \mathbb{N}$ *such that* $\phi_k(\alpha) = Q_k(\alpha) = 0$*, then* $\alpha = 0$.

3. The first order differential equation for the Carathéodory function

Let $u \in \Lambda'$ be a regular hermitian functional and $f \in \Lambda$. We define the linear functional $fu \in \Lambda'$ as

$$\langle fu, g(\xi)\rangle = \langle u, f(\xi)g(\xi)\rangle,\ g \in \Lambda,$$

and the derivative $\mathrm{D}u \in \Lambda'$ as

$$\langle \mathrm{D}\,u, f(\xi)\rangle = -i\langle \xi u, f'(\xi)\rangle = -i\langle u, \xi f'(\xi)\rangle.$$

In [2, 13, 18] it is established the equivalence between the Laguerre-Hahn affine character of a hermitian linear functional u, and the distributional equation

$$\mathrm{D}(Au) = Bu + zH\mathcal{L} \tag{12}$$

where $\mathcal{L}$ is the Lebesgue operator and A, B, H are polynomials. We remark that when $H = 0$ and $A \not\equiv 0$ in (12), u is said to be semi-classical.

In this section we study the relation between regular hermitian functionals u that satisfy (12) and a first order differential equation for the corresponding F.

We begin by establishing some properties for the function F. Throughout this section we will use the representation (5) for F.

Lemma 1. *If A and B are polynomials and u is a hermitian linear functional, the following relations hold, for $|z| \neq 1$:*

$$\langle B(e^{i\theta})u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = P(z) + B(z)F(z)\,, \tag{13}$$

$$A(z)F'(z) = -A'(z)F(z) + Q(z) + \frac{1}{zi}\langle D\,(Au), \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle \tag{14}$$

where P and Q are the polynomials defined by

$$P(z) = \langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\big(B(e^{i\theta}) - B(z)\big)\rangle \tag{15}$$

$$Q(z) = -A'(z) - \langle u_\theta, 2e^{i\theta}\sum_{k=2}^{\deg(A)}\frac{A^{(k)}(z)}{k!}(e^{i\theta}-z)^{k-2}\rangle \tag{16}$$

Proof: Since

$$\langle B(e^{i\theta})u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = \langle u, \frac{e^{i\theta}+z}{e^{i\theta}-z}\big(B(e^{i\theta}) - B(z)\big)\rangle + B(z)\langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle$$

and $\langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\big(B(e^{i\theta}) - B(z)\big)\rangle$ is a polynomial we get (13) with P defined as referred.

To obtain (14) we proceed as follows:

$$\begin{aligned} A(z)F'(z) \quad &= \langle u_\theta, \frac{2e^{i\theta}}{(e^{i\theta}-z)^2}A(z)\rangle \\ &= -\langle u_\theta, \frac{2e^{i\theta}}{(e^{i\theta}-z)^2}(A(e^{i\theta}) - A(z))\rangle + \langle u_\theta, \frac{2e^{i\theta}A(e^{i\theta})}{(e^{i\theta}-z)^2}\rangle \\ &= -\langle u_\theta, 2e^{i\theta}\sum_{k=1}^{\deg(A)}\frac{A^{(k)}(z)}{k!}(e^{i\theta}-z)^{k-2}\rangle + \langle u_\theta, \frac{2e^{i\theta}A(e^{i\theta})}{(e^{i\theta}-z)^2}\rangle \end{aligned}$$

But

$$\sum_{k=1}^{\deg(A)} \frac{A^{(k)}(z)}{k!}(e^{i\theta}-z)^{k-2} = \frac{A'(z)}{e^{i\theta}-z} + \sum_{k=2}^{\deg(A)} \frac{A^{(k)}(z)}{k!}(e^{i\theta}-z)^{k-2}$$

and, multiplying and dividing by z in the second term and taking into account that

$$\langle u_\theta, \frac{2ze^{i\theta}A(e^{i\theta})}{(e^{i\theta}-z)^2}\rangle = -\langle A(e^{i\theta})u_\theta, e^{i\theta}\frac{\partial}{\partial\theta}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\rangle,$$

we get

$$\begin{aligned} A(z)F'(z) = -A'(z)\left(F(z)+c_0\right) - \langle u_\theta, 2e^{i\theta}\sum_{k=2}^{\deg(A)} \frac{A^{(k)}(z)}{k!}(e^{i\theta}-z)^{k-2}\rangle \\ - \frac{1}{z}\langle A(e^{i\theta})u, e^{i\theta}\frac{\partial}{\partial\theta}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\rangle \end{aligned}$$

Now we use the definition of D and assume that $c_0 = 1$, to get

$$A(z)F'(z) = -A'(z)F(z) + Q(z) + \frac{1}{zi}\langle \mathrm{D}\,(Au), \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle,$$

with Q given by (16). ■

Remark . Throughout this section we will use the notation

$$P_{A,B}(z) = zQ(z) - iP(z)$$

where the polynomials P and Q are defined in terms of A and B by (15) and (16), respectively.

Next, we recover a result established in [2, 13], but here a different approach is used.

Theorem 2. *Let u be a regular hermitian functional. If u satisfies $D(Au) = Bu + \xi H(\xi)\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue functional, and A, B, H are polynomials, then F satisfies the first order differential equations*

$$zA(z)F'(z) = \left(-iB(z) - zA'(z)\right)F(z) + P_{A,B}(z) - 2izH(z), |z| < 1 \quad (17)$$

$$zA(z)F'(z) = \left(-iB(z) - zA'(z)\right)F(z) + P_{A,B}(z), \quad |z| > 1 \quad (18)$$

Conversely, if F satisfies (17) and (18), then u satisfies the functional equation

$$D(Au) = Bu + \xi H(\xi)\mathcal{L}.$$

Proof: Let u satisfy $\mathrm{D}\,(Au) = Bu + \xi H(\xi)\mathcal{L}$. By substituting $\mathrm{D}\,(Au) = Bu + \xi H(\xi)\mathcal{L}$ in (14) we obtain

$$A(z)F'(z) = -A'(z)F(z) + Q(z) + \frac{1}{iz}\langle Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle + \frac{1}{iz}\langle e^{i\theta}H(e^{i\theta})\mathcal{L}, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle$$

From (13) follows

$$\begin{aligned} &A(z)F'(z) \\ &\quad = -A'(z)F(z) + Q(z) + \frac{P(z)+B(z)F(z)}{iz} + \frac{1}{iz}\langle e^{i\theta}H(e^{i\theta})\mathcal{L}, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle \end{aligned} \quad (19)$$

Let $e^{i\theta}H(e^{i\theta}) = h_1 e^{i\theta} + \cdots + h_l(e^{i\theta})^l$, for some $l \in \mathbb{N}$.

Since, for $|z|<1$,

$$\begin{aligned} \langle e^{i\theta}H(e^{i\theta})\mathcal{L}, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle &= \langle (h_1 e^{i\theta} + \cdots + h_l(e^{i\theta})^l)\mathcal{L}, 1 + 2\sum_{k=1}^{+\infty}(e^{i\theta})^{-k}z^k\rangle \\ &= 2\left(h_1 z + h_2 z^2 + \cdots + h_l z^l\right), \\ &= 2zH(z) \end{aligned}$$

then, for $|z|<1$, (19) is equivalent to

$$A(z)F'(z) = -A'(z)F(z) + Q(z) + \frac{1}{zi}\left(P(z) + B(z)F(z)\right) + \frac{1}{iz}2zH(z)$$

and we obtain the equation

$$zA(z)F'(z) = \left(-iB(z) - zA'(z)\right)F(z) + C_1(z), \quad |z|<1,$$

with $C_1(z) = zQ(z) - iP(z) - 2izH(z) = P_{A,B}(z) - 2izH(z)$.

On the other hand, for $|z|>1$,

$$\langle e^{i\theta}H(e^{i\theta})\mathcal{L}, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = \langle (h_1 e^{i\theta} + \cdots + h_l(e^{i\theta})^l)\mathcal{L}, -1 - 2\sum_{k=1}^{+\infty}(e^{i\theta})^{k}z^{-k}\rangle = 0.$$

Therefore, for $|z|>1$, (19) is equivalent to

$$zA(z)F'(z) = \left(-iB(z) - zA'(z)\right)F(z) + C_2(z),$$

with $C_2(z) = zQ(z) - iP(z) = P_{A,B}(z)$.

Conversely, let F satisfy equations (17) and (18). We observe that, if F satisfies a differential equation with polynomial coefficients

$$zA(z)F'(z) = (-iB(z) - zA'(z))F(z) + C(z),$$

then, from (14) and (13), we obtain

$$\langle Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle - P(z) + iC(z) = izQ(z) + \langle \mathrm{D}\,(Au), \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle$$

and the following equation follows

$$\langle \mathrm{D}\,(Au) - Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = R(z) \tag{20}$$

where $R(z) = iC(z) - P(z) - izQ(z) = iC(z) - iP_{A,B}(z)$.

We now study equation (20) in both domains, $|z| < 1$ and $|z| > 1$:
a) since, if $|z| < 1$, equation (17) holds, then $C(z) = P_{A,B}(z) - 2izH(z)$, and (20) becomes

$$\langle \mathrm{D}\,(Au) - Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = 2zH(z)$$

Since $\dfrac{e^{i\theta}+z}{e^{i\theta}-z} = 1 + 2\sum_{k=1}^{+\infty}(e^{i\theta})^{-k}z^k, \ \ |z| < 1$, last equation is equivalent to

$$\langle \mathrm{D}\,(Au) - Bu, 1\rangle + 2\sum_{k=1}^{+\infty}\langle \mathrm{D}\,(Au) - Bu, (e^{i\theta})^{-k}\rangle z^k = 2zH(z) \tag{21}$$

b) since, if $|z| > 1$, equation (18) holds, then $C(z) = P_{A,B}(z)$ and (20) becomes

$$\langle \mathrm{D}\,(Au) - Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = 0$$

Since $\dfrac{e^{i\theta}+z}{e^{i\theta}-z} = -1 - 2\sum_{k=1}^{+\infty}(e^{i\theta})^{k}z^{-k}$, last equation is equivalent to

$$-\langle \mathrm{D}\,(Au) - Bu, 1\rangle - 2\sum_{k=1}^{+\infty}\langle \mathrm{D}\,(Au) - Bu, (e^{i\theta})^{k}\rangle z^{-k} = 0 \tag{22}$$

Finally, from (21) and (22), we have

$$\begin{aligned}
&\langle \mathrm{D}\,(Au) - Bu, (e^{i\theta})^{k}\rangle = 0, \ \forall k \geq 0\\
&\langle \mathrm{D}\,(Au) - Bu, (e^{i\theta})^{-k}\rangle = 0, \ \forall k > \deg(H) + 1\\
&\langle \mathrm{D}\,(Au) - Bu, (e^{i\theta})^{-k}\rangle = h_k, \ k = 1, \ldots, \deg(H) + 1
\end{aligned}$$

with $zH(z) = h_1 z + h_2 z^2 + \cdots + h_l z^l$.
Therefore, we obtain the functional equation $\mathrm{D}\,(Au) - Bu = \xi H(\xi)\mathcal{L}$. ∎

Note that if u is a semi-classical functional such that $\mathrm{D}\,(Au) = Bu$, then the function F associated with u satisfies a first order linear differential equation with polynomial coefficients,

$$zA(z)F'(z) = (-iB(z) - zA'(z)))F(z) + C(z),\ |z| \neq 1,$$

as is stated in [18].

Corollary 3. *Let F satisfy*

$$zA(z)F'(z) = (-iB(z) - zA'(z))F(z) + C(z),\ |z| \neq 1\,.$$

Then, the corresponding linear functional u is semi-classical and satisfies $D\,(Au) = Bu$ if and only if $C(z) = P_{A,B}(z)$.

Finally, we study the case of one differential equation for F. We will need the following lemma, which is a generalization of a result from [1].

Lemma 2. *Let u be a regular hermitian functional. If there exist polynomials A, B, H such that $D\,(Au) = Bu + H\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue functional, then the following equation holds,*

$$D\,(A + \overline{A})u = (B + \overline{B})u + (H + \overline{H})\mathcal{L} \tag{23}$$

Conversely, if (23) holds, then there exist polynomials $\tilde{A}, \tilde{B}, \tilde{H}$ such that $D\,(\tilde{A}u) = \tilde{B}u + \tilde{H}\mathcal{L}$.

Proof: If $\mathrm{D}\,(Au) = Bu + H\mathcal{L}$, then

$$\langle \mathrm{D}\,(Au), \xi^k \rangle = \langle Bu + H\mathcal{L}, \xi^k \rangle,\ \forall k \in \mathbb{Z}.$$

Applying conjugates, follows

$$\langle \mathrm{D}\,(\overline{A}u), \xi^{-k} \rangle = \langle \overline{B}u + \overline{H}\mathcal{L}, \xi^{-k} \rangle,\ \forall k \in \mathbb{Z}.$$

Therefore, we get

$$\langle \mathrm{D}\,((A + \overline{A})u), \xi^n \rangle = \langle (B + \overline{B})u + (H + \overline{H})\mathcal{L}, \xi^n \rangle,\ \forall n \in \mathbb{Z}$$

and (23) follows.
Conversely, if u satisfies (23), then

$$\langle \mathrm{D}\,((A + \overline{A})u), \xi^k \rangle = \langle (B + \overline{B})u, \xi^k \rangle + \langle (H + \overline{H})\mathcal{L}, \xi^k \rangle,\ \forall k \in \mathbb{Z}.$$

From the definition of D, we obtain for all $k \in \mathbb{Z}$,

$-ik\langle u, (A(\xi)+\overline{A}(1/\xi))\xi^k\rangle = \langle u, (B(\xi)+\overline{B}(1/\xi))\xi^k\rangle + \langle \mathcal{L}, (H(\xi)+\overline{H}(1/\xi))\xi^k\rangle.$

Let $s = \max\{\deg(A), \deg(B), \deg(H)\}$. Last equation can be written as

$$\begin{aligned} &-ik\langle u, \xi^s(A(\xi)+\overline{A}(1/\xi))\xi^{k-s}\rangle \\ &\quad = \langle u, \xi^s(B(\xi)+\overline{B}(1/\xi))\xi^{k-s}\rangle + \langle \mathcal{L}, \xi^s(H(\xi)+\overline{H}(1/\xi))\xi^{k-s}\rangle\,,\ k \in \mathbb{Z}\,. \end{aligned}$$

If we write $k = s+m$ and

$$\begin{aligned} A_1(\xi) &= \xi^s(A(\xi)+\overline{A}(1/\xi)) \\ B_1(\xi) &= \xi^s(B(\xi)+\overline{B}(1/\xi)) \\ H_1(\xi) &= \xi^s(H(\xi)+\overline{H}(1/\xi)) \end{aligned}$$

then A_1, B_1, H_1 are polynomials, and last functional equation is

$$-i(s+m)\langle u, A_1(\xi)\xi^m\rangle = \langle u, B_1(\xi)\xi^m\rangle + \langle \mathcal{L}, H_1(\xi)\xi^m\rangle$$

which is equivalent to

$$-im\langle u, A_1(\xi)\xi^m\rangle = \langle u, (B_1(\xi)+isA_1(\xi))\xi^m\rangle + \langle \mathcal{L}, H_1(\xi)\xi^m\rangle.$$

From the definition of D, follows

$$\langle \mathrm{D}\,(A_1 u), \xi^m\rangle = \langle (B_1+isA_1)u, \xi^m\rangle + \langle H_1\mathcal{L}, \xi^m\rangle,\ \forall m \in \mathbb{Z},$$

and we obtain the required result with $\tilde{A} = A_1,\ \tilde{B} = B_1 + isA_1,\ \tilde{H} = H_1$. ■

Theorem 3. *Let u be a regular hermitian functional. If F satisfies a first order differential equation with polynomial coefficients*

$$zA(z)F'(z) = (-iB(z) - zA'(z))\,F(z) + C(z),\ |z|<1 \tag{24}$$

then there exist polynomials $\tilde{A}, \tilde{B}, \tilde{H}$ such that u satisfies $D\,(\tilde{A}u) = \tilde{B}u + \tilde{H}\mathcal{L}$, with $\mathcal{L}$ the Lebesgue functional.

Proof: If F satisfies (24), following the same steps as in the second part of the proof of theorem 2 (see (20)), we obtain the equation

$$\langle \mathrm{D}\,(Au) - Bu, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = i(C(z) - P_{A,B}(z)),\ |z|<1 \tag{25}$$

Applying conjugates and the transformation $z \to 1/z$, follows

$$\langle \mathrm{D}\,(\overline{A}u) - \overline{B}u, \frac{e^{-i\theta}+1/z}{e^{-i\theta}-1/z}\rangle = -i(\overline{C}(1/z) - \overline{P}_{A,B}(1/z))\,.$$

Since $\dfrac{e^{-i\theta}+1/z}{e^{-i\theta}-1/z} = -\dfrac{e^{i\theta}+z}{e^{i\theta}-z}$, last equation is equivalent to

$$\langle \mathrm{D}\,(\overline{A}u) - \overline{B}u, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = i(\overline{C}(1/z) - \overline{P}_{A,B}(1/z)),\ |z| > 1 \tag{26}$$

Now, since there exists an analytic continuation outside the unit disk, in (25), and inside the unit disk, in (26), we get, for $1-\epsilon_2 < |z| < 1+\epsilon_1,\ \epsilon_1, \epsilon_2 > 0$,

$$\langle \mathrm{D}\,((A+\overline{A})u) - (B+\overline{B})u, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = i(C(z) - P_{A,B}(z)) + i(\overline{C}(1/z) - \overline{P}_{A,B}(1/z)).$$

By computing the moments of the hermitian functional $\mathrm{D}\,((A+\overline{A})u) - (B+\overline{B})u$, from last equation follows $\mathrm{D}\,((A+\overline{A})u) - (B+\overline{B})u = (H+\overline{H})\mathcal{L}$, with $H(\xi) = i(C(\xi) - P_{A,B}(\xi))/2$. From previous lemma, we obtain the required result. ∎

4. First order structure relations for orthogonal polynomials on the unit circle

In this section we establish the equivalence between the first order differential equation

$$zAF' + BF + C = 0,$$

for the Carathéodory function associated with a hermitian functional u, and first order structure relations for the corresponding orthogonal polynomials, the associated polynomials of the second kind and for the functions of the second kind. This will be done using the same ideas of [6].

Theorem 4. *Let u be a regular and hermitian functional, $\{\phi_n\}$ the sequence of monic orthogonal polynomials with respect to u, $\{\Omega_n\}$ the associated polynomials of the second kind and $\{Q_n\}$ the functions of the second kind. If there exist polynomials A, B, C such that F satisfies*

$$zA(z)F'(z) + B(z)F(z) + C(z) = 0,\ |z| < 1$$

then there exist polynomials G_n and H_n with degrees not depending on n, such that the following relations holds, for all $n \in \mathbb{N}$,

$$zA(z)\phi_n'(z) = (G_n(z) + \frac{B}{2}(z))\phi_n(z) + H_n(z)\phi_n^*(z) \tag{27}$$

$$zA(z)\Omega_n'(z) = (G_n(z) - \frac{B}{2}(z))\Omega_n(z) - H_n(z)\Omega_n^*(z) + C(z)\phi_n(z) \tag{28}$$

$$zA(z)Q_n'(z) = (G_n(z) - \frac{B}{2}(z))Q_n(z) - H_n(z)Q_n^*(z) \tag{29}$$

Conversely, if equations (27), (28) and (29) hold, for all $n \in \mathbb{N}$, then F satisfies a first order linear differential equation with polynomial coefficients

$$zAF' + BF + C = 0.$$

Proof: Before going into the proof we remark that if $\phi_n(0) = 0,\ \forall n \in \mathbb{N}$, then $\phi_n(z) = \Omega_n(z) = z^n,\ Q_n(z) = 2z^n,\ \phi_n^*(z) = \Omega_n^*(z) = 1,\ Q_n^*(z) = 0,\ F(z) = 1$, and the result holds with $A = 1, B = 0, C = 0$, and the differential relations (27), (28) and (29) with $G_n = n, H_n = 0,\ \forall n \in \mathbb{N}$. So, in what follows we will assume that we are not in the case $\phi_n(0) = 0,\ \forall n \in \mathbb{N}$.

Using (6) in $zAF' + BF + C = 0$ we get

$$zA\left\{\frac{\Omega_n\phi_n' - \Omega_n'\phi_n}{\phi_n^2}\right\} + zA\left(\frac{Q_n}{\phi_n}\right)' - B\frac{\Omega_n}{\phi_n} + B\frac{Q_n}{\phi_n} + C = 0.$$

Therefore the following equation holds,

$$zA\left(\Omega_n\phi_n' - \Omega_n'\phi_n\right) - B\Omega_n\phi_n + C\phi_n^2 = \Theta_n(z) \tag{30}$$

with

$$\Theta_n(z) = \left\{-zA(z)\left(\frac{Q_n(z)}{\phi_n(z)}\right)' - B(z)\frac{Q_n(z)}{\phi_n(z)}\right\}\phi_n^2(z) \tag{31}$$

From the asymptotic expansion of Q_n in $|z| < 1$ (see (10)), and since the left side of (30) is a polynomial, we get $\Theta_n = z^n\Theta_{n,1}$, for some polynomial $\Theta_{n,1}$. Moreover, using the asymptotic expansion of Q_n in $|z| > 1$ (see (11)), we conclude that $\Theta_{n,1}$ has bounded degree,

$$\deg(\Theta_{n,1}) = \max\{\deg(A) - 1, \deg(B) - 1\},\ \forall n \in \mathbb{N}.$$

Thus, (30) becomes

$$-\phi_n\left\{zA\Omega_n' + B\Omega_n - C\phi_n\right\} + \Omega_n(zA\phi_n') = z^n\Theta_{n,1}(z).$$

Using (8) follows

$$-\phi_n \left\{zA\Omega_n' + B\Omega_n - C\phi_n\right\} + \Omega_n(zA\phi_n') = \Theta_{n,2}(z)\left(\phi_n^*\Omega_n + \phi_n\Omega_n^*\right),$$

with $\Theta_{n,2}(z) = \Theta_{n,1}(z)/(2h_n)$, and we obtain

$$\phi_n \left\{zA\Omega_n' + \frac{B}{2}\Omega_n - C\phi_n + \Theta_{n,2}\Omega_n^*\right\} = \Omega_n \left\{zA\phi_n' - \frac{B}{2}\phi_n - \Theta_{n,2}\phi_n^*\right\} \quad (32)$$

We distinguish the following cases (see corollary 2):
a) ϕ_n and Ω_n have no common roots, $\forall n \in \mathbb{N}$, i.e., $\phi_n(0) \neq 0, \forall n \in \mathbb{N}$;
b) There exists a finite number of indexes $k \in \mathbb{N}$ such that ϕ_k and Ω_k have common roots, i.e., $\phi_k(0) = \Omega_k(0) = 0$ for a finite number of k's;
c) There exists $n_0 > 1$ such that $\phi_n(0) = 0$, $\forall n \geq n_0$.

Case a): If ϕ_n and Ω_n do not have common roots, then we conclude that there exists a polynomial l_n, $\forall n \in \mathbb{N}$, such that

$$\begin{cases} zA\phi_n' - \frac{B}{2}\phi_n - \Theta_{n,2}\phi_n^* = l_n\phi_n \\ zA\Omega_n' + \frac{B}{2}\Omega_n - C\phi_n + \Theta_{n,2}\Omega_n^* = l_n\Omega_n \end{cases} \quad (33)$$

and we obtain (27) and (28) with $G_n = l_n$ and $H_n = \Theta_{n,2}$. Moreover, as $\deg(H_n)$ is bounded, then $\deg(G_n)$ is bounded,

$$\deg(G_n) = \max\{\deg(A), \deg(B)\},\ \forall n \in \mathbb{N}.$$

Case b): We first suppose $\phi_1(0) \neq 0, \ldots, \phi_{k-1}(0) \neq 0$, and k is the first index such that $\phi_k(0) = 0$. Then, ϕ_n and Ω_n have no common roots, for $n = 1, \ldots, k-1$. From case a), equations (33) hold for $n = 1, \ldots, k-1$. Now we write equations (33) for $k-1$ and multiply by z, to obtain

$$\begin{cases} zAz\phi_{k-1}' - \frac{B}{2}z\phi_{k-1} - z\Theta_{k-1,2}\phi_{k-1}^* = l_{k-1}z\phi_{k-1} \\ zAz\Omega_{k-1}' + \frac{B}{2}z\Omega_{k-1} - Cz\phi_{k-1} + z\Theta_{k-1,2}\Omega_{k-1}^* = l_{k-1}z\Omega_{k-1} \end{cases}$$

By substituting

$$\phi_k(z) = z\phi_{k-1}(z),\ \phi_k^*(z) = \phi_{k-1}^*(z),\ z\phi_{k-1}'(z) = \phi_k'(z) - \phi_{k-1}(z)$$

and

$$\Omega_k(z) = z\Omega_{k-1}(z),\ \Omega_k^*(z) = \Omega_{k-1}^*(z),\ z\Omega_{k-1}'(z) = \Omega_k'(z) - \Omega_{k-1}(z)$$

in previous equations follows

$$\begin{cases} zA\phi_k' - \frac{B}{2}\phi_k - z\Theta_{k-1,2}\phi_k^* = (l_{k-1} + A)\phi_k \\ zA\Omega_k' + \frac{B}{2}\Omega_{k1} - C\phi_k + z\Theta_{k-1,2}\Omega_k^* = (l_{k-1} + A)\Omega_k \end{cases}$$

and we obtain (27) and (28) for $n = k$ with $G_k = l_{k-1} + A$ and $H_k = z\Theta_{k-1,2}$. Further, if $\phi_{k+1}(0) = \cdots = \phi_{k+k_0}(0) = 0$, and $\phi_{k+k_0+1}(0) \neq 0$ for some $k_0 \in \mathbb{N}$, using the same procedure as before, the differential relations (27) and (28) are obtained for $n = k+1, \ldots, k+k_0$, with

$$G_n = l_{k-1} + (n-k+1)A, \ H_n = z^{n-k+1}\Theta_{k-1,2}.$$

Case c): If $\phi_n(0) = 0$, $\forall n \geq n_0$, then ϕ_n and Ω_n are polynomials of the Bernstein-Szegő type,

$$\phi_n(z) = z^{n-n_0+1}\phi_{n_0-1}(z), \ \Omega_n(z) = z^{n-n_0+1}\Omega_{n_0-1}(z).$$

Applying the same procedure as before we conclude that equations (27) and (28) hold for $n \in \mathbb{N}$, and, $\forall n \geq n_0$, the polynomials G_n and H_n are given by

$$G_n = l_{n_0-1} + (n-n_0+1)A, \ H_n = z^{n-n_0+1}\Theta_{n_0-1,2}.$$

On the other hand, if we replace Θ_n by $2h_n z^n \Theta_{n,2}$ in (31) we get

$$\left\{-zA\left(\frac{Q_n}{\phi_n}\right)' - B\left(\frac{Q_n}{\phi_n}\right)\right\}\phi_n^2 = \Theta_{n,2}(z)2h_n z^n.$$

Using (9) we get

$$\left\{-zA\left(\frac{Q_n}{\phi_n}\right)' - B\left(\frac{Q_n}{\phi_n}\right)\right\}\phi_n^2 = \Theta_{n,2}(z)(\phi_n^* Q_n + \phi_n Q_n^*).$$

Therefore, $\forall n \in \mathbb{N}$,

$$\left\{zAQ_n' + \frac{B}{2}Q_n + \Theta_{n,2}Q_n^*\right\}\phi_n = \left\{zA\phi_n' - \frac{B}{2}\phi_n - \Theta_{n,2}\phi_n^*\right\}Q_n.$$

If we distinguish the two cases (see corollary 2):
a) ϕ_n and Q_n have no common roots, $\forall n \in \mathbb{N}$, i.e., $\phi_n(0) \neq 0, \forall n \in \mathbb{N}$;
b) ϕ_n and Q_n have the common root $z = 0$;
then, applying the same procedure as before, we conclude that, in both cases, there exists a polynomial L_n such that

$$\begin{cases} zA\phi_n' - \frac{B}{2}\phi_n - \Theta_{n,2}\phi_n^* = L_n\phi_n \\ zAQ_n' + \frac{B}{2}Q_n + \Theta_{n,2}Q_n^* = L_nQ_n. \end{cases}$$

Since $L_n = l_n$, we obtain (29) with $G_n = l_n$ and $H_n = \Theta_{n,2}$.

To prove the converse result we use (6) and (7) in equation (28), thus obtaining

$$zA\,(Q_n' - \phi_n' F - \phi_n F') = (G_n - \frac{B}{2})\,(Q_n - \phi_n F) - H_n\,(Q_n^* + \phi_n^* F) + C\phi_n,$$

i.e.,

$$\begin{aligned} &zAQ_n' + (\frac{B}{2} - G_n)Q_n + H_n Q_n^* \\ &\qquad = \left\{ zA\phi_n' - G_n\phi_n + \frac{B}{2}\phi_n - H_n\phi_n^* \right\} F + \{zAF' + C\}\,\phi_n. \end{aligned}$$

From (27) and (29) we obtain $\{zAF' + BF + C\}\,\phi_n = 0$, and $zAF' + BF + C = 0$ follows. ■

Remark . Moreover, from (7) and using the same reasoning as before, we deduce the equations for ϕ_n^* and Q_n^*, $\forall n \in \mathbb{N}$,

$$zA(\phi_n^*)' = (S_n + B/2)\phi_n^* - T_n\phi_n \tag{34}$$

$$zA(Q_n^*)' = T_n Q_n + (S_n - B/2)Q_n^* \tag{35}$$

where S_n, T_n are bounded degree polynomials.

From the differential equations (27), (29), (34) and (35), we obtain a differential systems for semi-classical orthogonal polynomials on the unit circle (analogue to the equations deduced in [7], for the real case).

Theorem 5. *Let $\{\phi_n\}$ be a sequence of monic orthogonal polynomials with respect to a semi-classical functional u, such that $D(Au) = Bu$. If u is positive definite and w is the absolutely continuous part of the corresponding measure, then the following equations hold,*

$$zA \begin{bmatrix} \phi_n & Q_n/w \\ \phi_n^* & -Q_n^*/w \end{bmatrix}' = \begin{bmatrix} G_n - \tilde{B}/2 & H_n \\ -T_n & S_n - \tilde{B}/2 \end{bmatrix} \begin{bmatrix} \phi_n & Q_n/w \\ \phi_n^* & -Q_n^*/w \end{bmatrix},\ \forall n \in \mathbb{N}$$

where $\tilde{B}(z) = -iB(z) - zA'$, and G_n, H_n, S_n, T_n are bounded degree polynomials.

Proof: If u satisfies $\mathrm{D}\,(Au) = Bu$ then the corresponding F satisfies $zAF' = \tilde{B}F + C$, with $\tilde{B} = -iB - zA'$, and C a polynomial (see corollary 3 of theorem 2).

From theorem 4 and the subsequent remark we have the following equations,

$$zA \begin{bmatrix} Q_n'/w \\ -(Q_n^*)'/w \end{bmatrix} = (\mathcal{B}_n + \frac{\tilde{B}}{2}\, I) \begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix} \tag{36}$$

and

$$zA \begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix}' = (\mathcal{B}_n - \frac{\tilde{B}}{2}\, I) \begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix} \tag{37}$$

with $\mathcal{B}_n = \begin{bmatrix} G_n & H_n \\ -T_n & S_n \end{bmatrix}$ and I the identity matrix of order two.

On the other hand, since $w'(z)/w(z) = \tilde{B}(z)/(zA(z))$, (see [18]) we obtain

$$zA \begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix}' = zA \begin{bmatrix} Q_n'/w \\ -(Q_n^*)'/w \end{bmatrix} - \tilde{B} \begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix} \tag{38}$$

Substituting (36) in (38) we get

$$zA \begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix}' = (\mathcal{B}_n - \frac{\tilde{B}}{2}\, I) \begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix} \tag{39}$$

Finally, from (37) and (39) we obtain the required differential system. ■

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