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RIEMANN-HILBERT PROBLEM ASSOCIATED WITH ANGELESCO SYSTEMS

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Abstract: Angelesco systems of measures with Jacobi type weights are considered. For such systems, strong asymptotic development expressions for sequences of associated Hermite-Padé approximants are found. In the procedure, an approach from Riemann-Hilbert Problem plays a fundamental role.

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1. The statement of the Riemann-Hilbert problem

Let $\Delta_j = [c_{1,j}, c_{2,j}] \subset \mathbb{R}$, $j = 1, 2$, be two intervals which are symmetric with respect to the origin. This means that $c_{1,1} = -c_{2,2}$ and $c_{1,2} = -c_{2,1}$. For each $j = 1, 2$, we take a holomorphic function h_j, on a neighborhood $\mathcal{V}_{h_j}$ of Δ_j , i.e. $h_j \in H(\mathcal{V}_{h_j})$. Let us define the system of measures (σ_1, σ_2) where σ_1 and σ_2 have the differential form

$$d\sigma_j(x) = \frac{h_j(x)dx}{\sqrt{(x-c_{1,j})(c_{2,j}-x)}}, \quad x \in \Delta_j\,, \quad j = 1, 2.$$

This system (σ_1, σ_2) belongs to the class of Angelesco systems introduced by Angelesco in [1]. Fix a multi-index $\boldsymbol{n} = (n_1, n_2)$, we say that a polynomial $Q_{\boldsymbol{n}} \not\equiv 0$ is a type II multiple-orthogonal polynomial corresponding to a system (σ_1, σ_2), if $\deg Q_{\boldsymbol{n}} \le |\boldsymbol{n}| = n_1+n_2$ and $Q_{\boldsymbol{n}}$ satisfies the following orthogonality conditions

$$\int_{\Delta_j} x^\nu Q_{\boldsymbol{n}}(x) d\sigma_j(x) = 0, \quad \nu = 0, \ldots, n_j - 1, \quad j = 1, 2.$$

It is well known that for any multi-index $\boldsymbol{n} = (n_1, n_2)$, the polynomial $Q_{\boldsymbol{n}}$ has exactly n_1+n_2 simple zeros lying in $\overset{\circ}{\Delta}_1 \cup \overset{\circ}{\Delta}_2$, where $\overset{\circ}{\Delta}_j$ denotes the interior set of Δ_j, $j = 1, 2$. Our propose in the present article consists in obtaining results about the strong asymptotic development of sequences of multi-orthogonal

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polynomials $\{Q_{\boldsymbol{n}} : \boldsymbol{n} \in \mathbb{Z}^2\}$. An effective method for such study with this kind of "very well" measures, is analyzing of the Riemann-Hilbert problem for multi-orthogonal polynomials, which was introduced in [5]. Let us consider a 3×3 square matrix, Y, whose entries are complex functions $Y_{s,k} : \mathbb{C} \to \mathbb{C}$, $s, k = 1, 2, 3$. Given a point $x \in \overset{\circ}{\Delta}_1 \cup \overset{\circ}{\Delta}_2$, the following matricial limits, where $z \in \mathbb{C} \setminus (\Delta_1 \cup \Delta_2)$ tending to x, represent the formal pontual limits of all entries of Y at the same time:

$$\begin{aligned} \lim_{z \to x} Y(z) &= Y_+(x), \quad \Im m\,(z) > 0 \\ \lim_{z \to x} Y(z) &= Y_-(x), \quad \Im m\,(z) < 0\,. \end{aligned}$$

Let $\delta_{s,k}$ denote the Kroneker delta function. Let us look for a matrix function Y which satisfies the following conditions:

(1) The entries of Y , $Y_{s,k}$, belongs to $H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$, which we write as $Y \in H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$;

(2) For each Δ_j, $j = 1, 2$, the so called jump condition takes place

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & \frac{2\pi i \delta_{1,j} h_1(x) dx}{\sqrt{(x-c_{1,1})(c_{2,1}-x)}} & \frac{2\pi i \delta_{2,j} h_2(x) dx}{\sqrt{(x-c_{1,2})(c_{2,2}-x)}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \overset{\circ}{\Delta}_j\,;$$

(3) Given a multi-index $\boldsymbol{n} = (n_1, n_2)$, we require the following asymptotic condition at infinity,

$$Y(z) \begin{pmatrix} z^{-|\boldsymbol{n}|} & 0 & 0 \\ 0 & z^{n_1} & 0 \\ 0 & 0 & z^{n_2} \end{pmatrix} = \mathbb{I} + \mathcal{O}\,(1/z) \quad \text{as} \quad z \to \infty,$$

where $\mathbb{I}$ is the identity matrix with rank 3 ;

(4) For each $i, j = 1, 2$, we set the following behavior around the end-points $c_{i,j}$,

$$Y(z) = \mathcal{O} \begin{pmatrix} 1 & \delta_{2,j} + \frac{\delta_{1,j}}{\sqrt{|z-c_{i,j}|}} & \delta_{1,j} + \frac{\delta_{2,j}}{\sqrt{|z-c_{i,j}|}} \\ 1 & \delta_{2,j} + \frac{\delta_{1,j}}{\sqrt{|z-c_{i,j}|}} & \delta_{1,j} + \frac{\delta_{2,j}}{\sqrt{|z-c_{i,j}|}} \\ 1 & \delta_{2,j} + \frac{\delta_{1,j}}{\sqrt{|z-c_{i,j}|}} & \delta_{1,j} + \frac{\delta_{2,j}}{\sqrt{|z-c_{i,j}|}} \end{pmatrix}.$$

This problem, which consists in finding the matrix function Y, was called in [5] a Riemann-Hilbert problem for type II multiple orthogonal polynomials, and for the system of measures (σ_1, σ_2), RHP in short. The solution Y is unique and has the form

$$Y(z) = \begin{pmatrix} Q_{\boldsymbol{n}}(z) & -\int_{\Delta_1} Q_{\boldsymbol{n}}(x) \frac{d\sigma_1(x)}{z-x} & -\int_{\Delta_2} Q_{\boldsymbol{n}} \frac{d\sigma_2(x)}{z-x} \\ d_1 Q_{\boldsymbol{n}_-^1}(z) & -\int_{\Delta_1} Q_{\boldsymbol{n}_-^1}(x) \frac{d\sigma_1(x)}{z-x} & -\int_{\Delta_2} Q_{\boldsymbol{n}_-^1} \frac{d\sigma_2(x)}{z-x} \\ d_2 Q_{\boldsymbol{n}_-^2}(z) & -\int_{\Delta_1} Q_{\boldsymbol{n}_-^2}(x) \frac{d\sigma_1(x)}{z-x} & -\int_{\Delta_2} Q_{\boldsymbol{n}_-^2} \frac{d\sigma_2(x)}{z-x} \end{pmatrix} \tag{1}$$

with $d_i^{-1} = -\int_{\Delta_i} x^{n_i-1} Q_{\boldsymbol{n}_-^i}(x) d\sigma_i(x)$, $i = 1, 2$, and if $\boldsymbol{n} = (n_1, n_2)$, $\boldsymbol{n}_-^1 = (n_1 - 1, n_2)$ and $\boldsymbol{n}_-^2 = (n_1, n_2 - 1)$.

The key of our procedure is based in finding the relationship between Y and a matrix function R which is the solution of another RHP with the following formulation. Suppose that γ is a closed simple and smooth contour on the complex plane $\mathbb{C}$, then find a matrix function, R, such that:

(1) $R : \mathbb{C} \to \mathbb{C}^{3\times 3}$ belongs to $H(\mathbb{C} \setminus \gamma)$;
(2) $R_+(\xi) = R_-(\xi) V_{\boldsymbol{n}}(\xi)$, $\xi \in \gamma$;
(3) $R(z) \to \mathbb{I}$ as $z \to \infty$,

where $V_{\boldsymbol{n}}$ is a 3×3 matrix function, which is called the jump matrix.

Given an arbitrary 3×3 matrix function $K = [K_{s,k}]_{s,k}$, $s, k = 1, 2, 3$, defined on a open set $\Omega \subset \mathbb{C}$ let us denote by $\|K\|$ (respectively, $\|K\|_\Omega$) the matrix infinity norm which consists in the maximum sum of row's entries modulus, defined for 3×3 matrices, i.e.

$$\|K\| = \max_{s=1,2,3} \sum_{k=1}^{3} |K_{s,k}|, \quad \text{(respectively,} \quad \|K\|_\Omega = \sup_\Omega \|K\|) .$$

Theorem 1 (See Theorem 3.1 in [7]). *Suppose that Ω is an open set containing γ. In condition (2) of the RHP for R, let us require for $V_{\boldsymbol{n}} \in H(\Omega)$ that there exist constants C and $\delta_{\boldsymbol{n}} > 0$ for which*

$$\|V_{\boldsymbol{n}} - \mathbb{I}\|_\Omega < \delta_{\boldsymbol{n}}.$$

Then, any solution of the RHP for R satisfies that

$$\|R(z) - \mathbb{I}\| < C \|V_{\boldsymbol{n}} - \mathbb{I}\|_\Omega \quad \textit{for every} \quad z \in \mathbb{C} \setminus \gamma.$$

Notice that if we know the relationship between R and Y and if we can also describe the development of R when $|\boldsymbol{n}| \to \infty$, we would have a description for the development of all entries of Y when $|\boldsymbol{n}| \to \infty$, particularly for $Y_{1,1}(z) = Q_{\boldsymbol{n}}(z)$.

The RHP for Y is not normalized in the sense that the conditions (3) at infinity for Y and R are different. In order to normalize the RHP, we are going to modify Y in such a way that we set another RHP with the same contours (possibly different jump conditions), for which the solution tends to the identity matrix as $z \to \infty$. For normalizing we need to take into account the behavior of $Y(z)$ for large z. This behavior depends on the distribution of the zeros of the multiple-orthogonal polynomials. The zero distribution of the orthogonal polynomials is usually given by an extremal problem in logarithmic potential theory. In section 2 we introduce some concepts and results which we will need about this theory and we will normalized the Riemann-Hilbert problem at infinity. In section 3 such a Riemann-Hilbert problem with oscillatory and exponentially decreasing jumps can be analyzed by using the steepest descent method introduced by Deift and Zhou (see [3, 4]).

2. The equilibrium problem and the normalization at infinity

Let us fix $j \in \{1, 2\}$. $\mathcal{M}_{1/2}(\Delta_j)$ denotes the set of all finite Borel measures whose supports, i.e. $\operatorname{supp}(\cdot)$, are contained in Δ_j with total variation $1/2$. Take $\mu_j \in \mathcal{M}_{1/2}(\Delta_j)$ and define its logarithmic potential as follows

$$V^{\mu_j}(z) = \int \log \frac{1}{|z-x|} d\mu_j(x), \qquad z \in \mathbb{C}.$$

For each pair of measures (μ_1, μ_2), where $\mu_j \in \mathcal{M}_{1/2}(\Delta_j)$, $j = 1, 2$, we define the quantities

$$m_j(\mu_1, \mu_2) = \min_{x \in \Delta_j} \left(2V^{\mu_j}(x) + V^{\mu_k}(x) \right), \quad j, k = 1, 2, \quad j \neq k.$$

The following Proposition is deduced immediately from the results of [6].

Proposition 1. *There exists a unique pair* $(\bar{\mu}_1, \bar{\mu}_2) \in \mathcal{M}_{1/2}(\Delta_1) \times \mathcal{M}_{1/2}(\Delta_2)$, *which satisfies for* $j, k = 1, 2$

$$2V^{\bar{\mu}_j}(x) + V^{\bar{\mu}_k}(x) = m_j(\bar{\mu}_1, \bar{\mu}_2) = m_j, \quad x \in \operatorname{supp}(\bar{\mu}_j) = \Delta_j, \quad j \neq k.$$

For each $j = 1, 2$ *the measure* $\bar{\mu}_j$ *has the following differential form*

$$d\bar{\mu}_1(x) = \frac{\rho_1(x)dx}{\sqrt{(x - c_{1,1})(c_{2,1} - x)}}, \qquad d\bar{\mu}_2(x) = \frac{\rho_2(x)dx}{\sqrt{(x - c_{1,2})(c_{2,2} - x)}},$$

where $\rho_j \in H(\mathcal{V}_{\rho_j})$, *with* $\mathcal{V}_{\rho_j}$ *denoting an open set which contains* Δ_j.

The pair $(\bar{\mu}_1, \bar{\mu}_2)$ is called extremal or equilibrium pair of measures with respect to (Δ_1, Δ_2). Let us denote for each $j = 1, 2$ the analytic potentials

$$g_j(z) = \int_{\Delta_j^*} \log(z-x)d\bar{\mu}_j(x) = -V^{\bar{\mu}_j}(z) + i\int_{\Delta_j^*} \arg(z-x)d\mu_j(x)\,,$$

where Δ_j^* is the support of the extremal measure, $\bar{\mu}_j$, that coincides in our case with Δ_j, for $j = 1, 2$. Substituting the potential logarithmic in Proposition 1 we obtain for each $j, k = 1, 2$ with $j \neq k$ that

$$-\left[g_{j+} + g_{j-}\right](x) - g_{k-}(x) = m_j\,, \quad x \in \Delta_j\,.$$

Observe that

$$\left(g_{j+} - g_{j-}\right)(x) = \begin{cases} 0 & \text{if} \quad c_{2,j} \leq x \\ i\pi & \text{if} \quad c_{1,j} \geq x \\ 2i\pi \int_x^{c_{2,j}} d\bar{\mu}_j(t) & \text{if} \quad x \in \Delta_j \end{cases}.$$

In what follows all the multi-indices will have the form $\boldsymbol{n} = (n, n)$. Let us introduce the matrices

$$G(z) = \begin{pmatrix} e^{-2n(g_1(z)+g_2(z))} & 0 & 0 \\ 0 & e^{2ng_1(z)} & 0 \\ 0 & 0 & e^{2ng_2(z)} \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-2nm_1} & 0 \\ 0 & 0 & e^{-2nm_2} \end{pmatrix}. \quad (2)$$

We define the matrix function $T = LYGL^{-1}$, where L, G are as in (2) and Y is given by (1). Hence T is the unique solution of the RHP:

(1) $T \in H(\mathbb{C} \setminus (\overset{\circ}{\Delta}_1 \cup \overset{\circ}{\Delta}_2))$;
(2) $T_+(x) = T_-(x)M(x)$, $x \in \Delta_1 \cup \Delta_2$;
(3) $T(z) = \mathbb{I} + \mathcal{O}(1/z)$ as $z \to \infty$;
(4) T and Y have the same behavior on the endpoints of the intervals Δ_j, for $j = 1, 2$,

where the jump matrix M has the form

$$M(x) = \begin{pmatrix} e^{-2ni\pi \int_x^{c_{2,j}} d\bar{\mu}_j(t)} & 2\delta_{j,1}\pi i w_1(x) & 2\delta_{j,2}\pi i w_2(x) \\ 0 & e^{2n\delta_{j,1} i\pi \int_x^{c_{2,1}} d\bar{\mu}_1(t)} & 0 \\ 0 & 0 & e^{2n\delta_{j,2} i\pi \int_x^{c_{2,2}} d\bar{\mu}_2(t)} \end{pmatrix}, \quad x \in \overset{\circ}{\Delta}_j\,.$$

3. The opening of the lens

For each $j=1,2$, let ϕ_j denote the function defined by

$$\phi_j(z)=i\pi\int_z^{c_{2,j}} d\bar{\mu}_j(t) \quad \text{for} \quad z\in\mathcal{V}_j=\mathcal{V}_{\rho_j}(\Delta_j^*)\cap\mathcal{V}_{h_j}(\Delta_j)\,.$$

Notice that $\phi_{j+}(x)=i\pi\int_x^{c_{2,j}} d\bar{\mu}_j(t)$ is purely imaginary and its derivative

$$\phi'_{j+}(x)=-i\pi\frac{\rho_j(x)}{\sqrt{(x-c_{1,j})(c_{2,j}-x)}}\,,$$

where $-\pi\rho_j(x)/\sqrt{(x-c_{1,j})(c_{2,j}-x)}<0\,,\quad x\in\Delta_j\,.$
Rewrite $\phi_j(z)=U_j(z)+iV_j(z)\in H(\mathcal{V}_j)$. By the Cauchy-Riemann conditions we have that the real part of ϕ_j, $\Re e\,\phi_j$, is an increasing function on any point $z\in\mathcal{V}_j$ with $\Im m\,(z)>0$. Since $\Re e\,\phi_j$ is zero in Δ_j , it is positive in such point. Notice $\phi_{j+}(x)=-\phi_{j-}(x)\,,\ x\in\overset{\circ}{\Delta}_j$, hence we can proceed analogously when $\Im m\,(z)<0$.

We analyze the jump function in Δ_j , i.e.

$$M(x)=\begin{pmatrix} e^{-2n\phi_{j+}(x)} & 2\pi i\delta_{j,1}w_1(x) & 2\pi i\delta_{j,2}w_2(x)\\ 0 & e^{-2\delta_{j,1}n\phi_{1-}(x)} & 0\\ 0 & 0 & e^{-2\delta_{j,2}n\phi_{2-}(x)}\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ -\frac{\delta_{j,1}e^{-2n\phi_{1-}(x)}}{2\pi i w_1(x)} & 1 & 0\\ -\frac{\delta_{j,2}e^{-2n\phi_{2-}(x)}}{2\pi i w_2(x)} & 0 & 1\end{pmatrix}$$

$$\times\begin{pmatrix}0 & 2\pi i\delta_{j,1}w_1(x) & 2\pi i\delta_{j,2}w_2(x)\\ -\frac{\delta_{1,j}}{2\pi i w_1(x)} & \delta_{j,2} & 0\\ -\frac{\delta_{2,j}}{2\pi i w_2(x)} & 0 & \delta_{j,1}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ -\frac{\delta_{j,1}e^{-2n\phi_{1+}(x)}}{2\pi i w_1(x)} & 1 & 0\\ -\frac{\delta_{j,2}e^{-2n\phi_{2+}(x)}}{2\pi i w_2(x)} & 0 & 1\end{pmatrix}.$$

Now we are going to follow an analogous procedure as in section 9 of [7]. Let us fix a $\delta>0$ such that the intervals $[c_{1,j}-\delta,c_{1,j}]$ and $[c_{2,j},c_{2,j}+\delta]$ are subsets of $\mathcal{V}_j$, $j=1,2$, and for each interval let us define two curves Σ_{j+}, Σ_{j-} in $\mathcal{V}_j$, which goes from $c_{1,j}$ to $c_{2,j}$, where $\Sigma_{j\pm}=[c_{1,j}-\delta,c_{1,j}]\cup\Sigma_{j\pm}^*\cup[c_{2,j},c_{2,j}+\delta]$, with the elements of the curves $\Sigma_{j\pm}^*$ satisfying that if $z\in\Sigma_{j\pm}^*$, then $0<\pm\Im m\,(z)$ (cf. Figure 1). Set $\Gamma_{j\pm}$ the domains that lie between $\Sigma_{j\pm}$ and Δ_j . Let us introduce the matrix function S, defined by

$$S(z)=T\begin{pmatrix}1 & 0 & 0\\ \mp\frac{\delta_{1,j}e^{-2n\phi_{1-}(z)}}{2\pi i w_1(z)} & 1 & 0\\ \mp\frac{\delta_{2,j}e^{-2n\phi_{2-}(z)}}{2\pi i w_2(z)} & 0 & 1\end{pmatrix}\,,\quad z\in\Gamma_{j\pm},\quad \text{and } S\equiv T,\quad \text{outside.} \tag{3}$$

Hence on open intervals $]c_{1,j}-\delta, c_{1,j}[$ and $]c_{2,j}, c_{2,j}+\delta[$ there are two combined jumps. For the function w_j we have that

$$w_{j+}(x) = -w_{j-}(x)\,, \quad x \in \mathbb{R} \cap \mathcal{V}(\Delta_j) \setminus \Delta_j\,, \quad j = 1, 2\,.$$

Observe that

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_{j,1} e^{-2n\phi_1}}{2\pi i w_{1-}} & 1 & 0 \\ \frac{\delta_{j,2} e^{-2n\phi_2}}{2\pi i w_{2-}} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \frac{\delta_{j,1} e^{-2n\phi_1}}{2\pi i w_{1+}} & 1 & 0 \\ \frac{\delta_{j,2} e^{-2n\phi_2}}{2\pi i w_{2+}} & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \left(\frac{1}{w_{1-}} + \frac{1}{w_{1+}}\right) \frac{\delta_{1,j} e^{-2n\phi_1}}{2\pi i} & 1 & 0 \\ \left(\frac{1}{w_{2-}} + \frac{1}{w_{2+}}\right) \frac{\delta_{2,j} e^{-2n\phi_1}}{2\pi i} & 0 & 1 \end{pmatrix} = \mathbb{I},$$

which means that S, defined by (3), is analytic function across $]c_{1,j} - \delta, c_{1,j}[$ and $]c_{2,j}, c_{2,j} + \delta[$, $j = 1, 2$. Let γ_j, $j = 1, 2$, be closed contours with the clockwise direction, such that for each $j = 1, 2$, $\gamma_j = \Sigma^*_{j-} \cup \Sigma^*_{j+}$. We have changed the direction of the curve Σ^*_{j+}. The function S satisfies the RHP:

(1) $S \in H(\mathbb{C} \setminus \cup_{j=1,2}(\Delta_j \cup \gamma_j))$;
(2) The jump conditions for $j = 1, 2$ are,

$$\begin{aligned} S_+(x) &= S_-(x) \begin{pmatrix} 0 & 2\delta_{1,j}\pi i w_1(x) & 2\delta_{2,j}\pi i w_2(x) \\ -\frac{\delta_{1,j}}{2\pi i w_1(x)} & \delta_{2,j} & 0 \\ -\frac{\delta_{2,j}}{2\pi i w_2(x)} & 0 & \delta_{1,j} \end{pmatrix} \quad \text{if } x \in \mathring{\Delta}_j\,, \\ S_+(z) &= S_-(z) \begin{pmatrix} 1 & 0 & 0 \\ \frac{\pm\delta_{1,j} e^{-2n\phi_1(z)}}{2\pi i w_1(z)} & 1 & 0 \\ \frac{\pm\delta_{2,j} e^{-2n\phi_2(z)}}{2\pi i w_2(z)} & 0 & 1 \end{pmatrix} \quad \text{if } z \in \gamma_j \cap \{\pm \Im m\, z < 0\}\,; \end{aligned}$$

(3) $S(z) = \mathbb{I} + \mathcal{O}(1/z)$ as $z \to \infty$;
(4) The conditions for the endpoints are the same as for T.

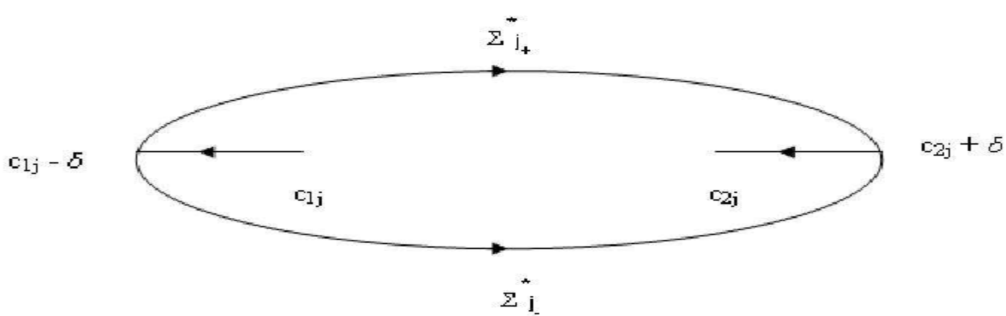

FIGURE 1. Opening of lens

Now, we consider the limiting problem, because for the matrix S the jump matrix function on each γ_j for $j=1,2$ tends to the identity matrix when $|\boldsymbol{n}|\to\infty$. We look for the matrix function N which satisfies the following RHP:

(1) $N\in H(\mathbb{C}\setminus(\Delta_1\cup\Delta_2))$;

(2) The jump conditions in $\overset{\circ}{\Delta}_j$ for $j=1,2$ are,

$$N_+(x)=N_-(x)\begin{pmatrix}0 & 2\delta_{1,j}\pi i w_1(x) & 2\delta_{2,j}\pi i w_2(x)\\ -\frac{\delta_{1,j}}{2\pi i w_1(x)} & \delta_{2,j} & 0\\ -\frac{\delta_{2,j}}{2\pi i w_2(x)} & 0 & \delta_{1,j}\end{pmatrix}; \tag{4}$$

(3) $N(z)=\mathbb{I}+\mathcal{O}(1/z)$ as $z\to\infty$;

(4) N satisfies the same conditions for the endpoints as S.

Set $N=KD$, where D is a diagonal matrix function and $K=[K_{k,l}]_{k,l}$, $k,l=1,2$, is the solution of the RHP:

(1) $K\in H(\mathbb{C}\setminus(\Delta_1\cup\Delta_2))$;

(2) The jump conditions in $\overset{\circ}{\Delta}_j$ for $j=1,2$ are, because of (4),

$$K_+(x)=K_-(x)\begin{pmatrix}0 & \frac{2\delta_{1,j}\pi i}{\sqrt{(c_{2,1}-x)(x-c_{1,1})}} & \frac{2\delta_{2,j}\pi i}{\sqrt{(c_{2,2}-x)(x-c_{1,2})}}\\ \frac{-\delta_{1,j}\sqrt{(c_{2,1}-x)(x-c_{1,1})}}{2\pi i} & \delta_{2,j} & 0\\ \frac{-\delta_{2,j}\sqrt{(c_{2,2}-x)(x-c_{1,2})}}{2\pi i} & 0 & \delta_{1,j}\end{pmatrix}; \tag{5}$$

(3) $K(z)=\mathbb{I}+\mathcal{O}(1/z)$ as $z\to\infty$;

(4) K and N have the same conditions for the endpoints.

Analogously to the ideas in [2], let us choose the branches of the square root which glue along the intervals Δ_j, $j=1,2$, i.e.

$$\left(\sqrt{(x-c_{2,j})(x-c_{1,j})}\right)_+=-\left(\sqrt{(x-c_{2,j})(x-c_{1,j})}\right)_-,\quad x\in\overset{\circ}{\Delta}_j,\ j=1,2.$$

For each $i=1,2,3$, we rewrite (5) as

$$\begin{cases}\left(\frac{\sqrt{(z-c_{2,1})(z-c_{1,1})}}{2\pi}K_{i,2}(z)\right)_\pm(x)=(K_{i,1}(z))_\mp(x)\\ (K_{i,3})_+(x)=(K_{i,3})_-(x)\end{cases},\quad x\in\overset{\circ}{\Delta}_1$$

$$\begin{cases} \left(\frac{\sqrt{(z-c_{2,2})(z-c_{1,2})}}{2\pi} K_{i,3}(z)\right)_{\pm}(x) = (K_{i,1}(z))_{\mp}(x) \\ (K_{i,2})_+(x) = (K_{i,2})_-(x) \end{cases}, \quad x \in \overset{\circ}{\Delta}_2$$

and we denote

$$\psi_0^i(z) = K_{i,1}(z)\,, \quad \text{and} \quad \psi_j^i(z) = \frac{\sqrt{(z-c_{2,j})(z-c_{1,j})}}{2\pi} K_{i,j+1}(z)\,, \ j = 1,2\,.$$

Then from the relations (5), we may interpret each row $i = 1,2,3$ of such matrix K as a function defined on a Riemann surface. Let $\mathcal{R}$ define the Riemann surface which has two cuts. One of them connects the two branch points $c_{1,1}$ and $c_{2,1}$ with the cut in the interval $\Delta_1 = [c_{1,1}, c_{2,1}]$. The other cut is made in the interval $\Delta_2 = [c_{1,2}, c_{2,2}]$, to connect the two other branch points $c_{1,2}$ and $c_{2,2}$. The sheet $\mathcal{R}_0$ is glued to another sheet $\mathcal{R}_1$ along the cut Δ_1, and $\mathcal{R}_0$ is also glued to $\mathcal{R}_2$ along the interval Δ_2. Let us denote by ψ^i, $i = 1,2,3$, three multi-valued function $\psi^i = \left(\psi_0^i, \psi_1^i, \psi_2^i\right)$, such that its components ψ_l^i, $i = 1,2,3$, $l = 0,1,2$, map the corresponding sheet $\mathcal{R}_l$ on $\mathbb{C}$, and satisfy for $j = 1,2$

$$\psi_{0\pm}^i(x) = \psi_{j\mp}^i(x), \quad x \in \overset{\circ}{\Delta}_j, \quad \psi_j^i(z) = \mathcal{O}(1) \quad \text{as} \quad z \to c_{k,j}\,, \quad k = 1,2\,; \quad (6)$$

for $j,k = 1,2$, $\psi_0^i(x) = \mathcal{O}(1)$, as $z \to c_{k,j}$; around the infinity, and $j = 1,2$,

$$\psi_0^i(z) = \delta_{i,1} + \mathcal{O}(1/z)\,, \quad \psi_j^i(z) = z\delta_{i,j+1} + \mathcal{O}(1) \quad \text{as} \quad z \to \infty\,.$$

The equalities (6) are equivalent to

$$(\psi_0^i\psi_j^i)_+(x) = (\psi_0^i\psi_j^i)_-(x)\,, \quad x \in \overset{\circ}{\Delta}_j\,, \quad j = 1,2\,.$$

From Liouville's theorem, it is easy to see that $(\psi_0^i\psi_1^i\psi_2^i)(z) \equiv 1$, $z \in \overline{\mathbb{C}}$. This implies that ψ_l^i, $l = 0,1,2$, $i = 1,2,3$, do not become zero. Hence

$$(\psi_0^i\psi_k^i)(z) = \frac{1}{\psi_l^i(z)} \in H\left(\overline{\mathbb{C}} \setminus \Delta_l\right), \quad l,k = 1,2, \quad l \neq k, \quad i = 1,2,3.$$

We obtain that for each $l = 1,2$, that is $x \in \overset{\circ}{\Delta}_l$,

$$\psi_{0+}^i(x)\psi_k^i(x) = \frac{1}{\psi_{l+}^i(z)} \quad \text{or equivalently} \quad (\psi_{l+}^i\psi_{l-}^i)(x) = \frac{1}{\psi_k^i(x)}.$$

That yields to the problems for ψ_l^i, $l = 1,2$:

- $\psi_l^i \in H(\overline{\mathbb{C}} \setminus \Delta_l)$;

- $(\psi^i_{l+}\psi^i_{l-})(x) = 1/\psi^i_k(x), \quad x \in \mathring{\Delta}_l\,;$
- $\psi^i_l(z) = z\delta_{i,l+1} + \mathcal{O}\left(1\right)$ as $z \to \infty\,;$
- $\psi^i_l(z) = \mathcal{O}\left(1\right)$ as $z \to c_{k,l},\ k = 1, 2\,.$

This problem is equivalent to the system of integral equations:

$$\psi^i_0(z) \;=\; \frac{1}{\psi^i_1(z)\psi^i_2(z)}$$

$$\psi^i_l(z) = \exp\left(\frac{\sqrt{(z-c_{1,l})(z-c_{2,l})}}{2\pi}\int_{\Delta_l}\frac{\log\psi^i_k(x)}{\sqrt{(x-c_{1,l})\,(c_{2,l}-x)}}\frac{dx}{z-x} + \delta_{i,l+1}g_{\Delta_l}(z)\right),$$

for $l = 1, 2$, with $z \in \overline{\mathbb{C}} \setminus \mathring{\Delta}_l$, and g_{Δ_j} is the analytic function which tends to ∞ as $\log z$, and whose real part vanishes in Δ_j, $j = 1, 2$

Let us find the diagonal 3×3 matrix function D with diagonal elements D_0, D_1, D_2, such that $N(z) \equiv K(z)D(z)$. The conditions (5) yield that entries of D must satisfy the following conditions

$$\begin{cases} h_j(x)D_{0\pm}(x) = D_{j\mp}(x) \\ D_{k+}(x) = D_{k-}(x) \end{cases} \quad \text{when} \quad x \in \mathring{\Delta}_j, \quad j, k = 1, 2, \quad k \neq j.$$

Analogously to the function ψ^i_l, we obtain the following problem for the entries of D:

i) $(D_0D_1D_2) \equiv 1$, which implies that for each $l = 0, 1, 2$, D_l does not become zero;
ii) $D_l \in H(\overline{\mathbb{C}} \setminus \Delta_l)$, $l = 1, 2\,;$
iii) $(D_{l+}D_{l-})(x) = h_s(x)/(D_s(x)), \quad s = 1, 2, \quad s \neq l, \quad x \in \mathring{\Delta}_l,\ l = 1, 2\,;$
iv) $D_l(z) = 1 + \mathcal{O}\left(1/z\right)$ as $z \to \infty$, $l = 1, 2$,
v) $D_l(z) = \mathcal{O}\left(1\right)$ as $z \to c_{k,l}$, $k = 1, 2$, $l = 1, 2\,.$

This problem is equivalent to the following system of integral equations where $l = 1, 2$,

$$D_l(z) \;=\; \exp\left(\frac{\sqrt{(z-c_{1,l})(z-c_{2,l})}}{2\pi}\int_{\Delta_l}\frac{\log\left(\frac{D_k(x)}{h_k(x)}\right)}{\sqrt{(x-c_{1,l})\,(c_{2,l}-x)}}\frac{dx}{z-x}\right)$$

$$D_0(z) \;=\; \frac{1}{D_1(z)D_2(z)} \quad \text{with} \quad z \in \overline{\mathbb{C}} \setminus \mathring{\Delta}_l\,.$$

Let us take the three multi-valued function $(D_0(z), D_1(z), D_2(z))$. Notice that for each $l = 0, 1, 2$, the function D_l is another function which maps the sheet $\mathcal{R}_l$ on $\mathbb{C}$. In our case the components of functions $(D_0(z), D_1(z), D_2(z))$ satisfy the conditions iv) and v) required for D_l, $l = 0, 1, 2$. Finally the matrix function N has the form

$$N(z) = \begin{pmatrix} (D_0\psi_0^1)(z) & \frac{(D_1\psi_1^1)(z)}{\sqrt{(x-c_{1,1})(c_{2,1}-x)}} & \frac{(D_2\psi_2^1)(z)}{\sqrt{(x-c_{1,2})(c_{2,2}-x)}} \\ (D_0\psi_0^2)(z) & \frac{(D_1\psi_1^2)(z)}{\sqrt{(x-c_{1,1})(c_{2,1}-x)}} & \frac{(D_2\psi_2^2)(z)}{\sqrt{(x-c_{1,2})(c_{2,2}-x)}} \\ (D_0\psi_0^3)(z) & \frac{(D_1\psi_1^3)(z)}{\sqrt{(x-c_{1,1})(c_{2,1}-x)}} & \frac{(D_2\psi_2^3)(z)}{\sqrt{(x-c_{1,2})(c_{2,2}-x)}} \end{pmatrix}. \tag{7}$$

We define $R(z) = S(z)N^{-1}(z)$. Since S and N have the same jump across $\mathring{\Delta}_j$, $j = 1, 2$, hence $R_+(x) = R_-(x)$ for $x \in \mathring{\Delta}_j$, $j = 1, 2$. From the definition of R, and the endpoint conditions for N, we can also deduce that $c_{i,k}$, $i, k = 1, 2$, are a removable singularity. Hence R is an analytic function across the full intervals Δ_1 and Δ_2, and it has jumps on the curve γ. Then we have the following RHP for R:

(1) $R \in H(\mathbb{C} \setminus (\gamma_1 \cup \gamma_2))$;

(2) The jump conditions are for $j = 1, 2$

$$R_+(z) = R_-(z) \begin{pmatrix} 1 & 0 & 0 \\ \pm\frac{\delta_{j,1} e^{-2n_1\phi_1(z)}}{2\pi i w_1(z)} & 1 & 0 \\ \pm\frac{\delta_{j,2} e^{-2n_2\phi_2(z)}}{2\pi i w_2(z)} & 0 & 1 \end{pmatrix} \quad \text{if} \quad z \in \gamma_j \cap \{\pm \Im m\, z < 0\};$$

(3) $R(z) = \mathbb{I} + \mathcal{O}(1/z)$.

Observe that matrix function R satisfies the hypothesis of the Theorem 1. Then uniformly for $z \in \mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$, we have that $R(z) = \mathbb{I} + \mathcal{O}\left(e^{c|\boldsymbol{n}|}\right)$, with $c > 0$ and

$$Y(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{|\boldsymbol{n}|m_1} & 0 \\ 0 & 0 & e^{|\boldsymbol{n}|m_2} \end{pmatrix} \left(\mathbb{I} + \mathcal{O}\left(e^{-c|\boldsymbol{n}|}\right)\right)$$
$$\times N(z) \begin{pmatrix} e^{|\boldsymbol{n}|(g_1+g_2)}(z) & 0 & 0 \\ 0 & e^{-|\boldsymbol{n}|(m_1+g_1(z))} & 0 \\ 0 & 0 & e^{-|\boldsymbol{n}|(m_2+g_2)(z)} \end{pmatrix},$$

where N is given by (7).

Finally, we state the main result of this paper.

Theorem 2.

$$Y_{1,1}(z) = Q_{\boldsymbol{n}}(z) = D_0(z)\psi_0^1(z)e^{|\boldsymbol{n}|(g_1+g_2)(z)}\left(1+\mathcal{O}\left(e^{-c|\boldsymbol{n}|}\right)\right),$$

as $|\boldsymbol{n}| \to \infty$*.*

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