RIEMANN-HILBERT PROBLEM ASSOCIATED WITH ANGELESCO SYSTEMS

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ABSTRACT: Angelesco systems of measures with Jacobi type weights are considered. For such systems, strong asymptotic development expressions for sequences of associated Hermite-Padé approximants are found. In the procedure, an approach from Riemann-Hilbert Problem plays a fundamental role.

AMS Subject Classification (2000): Primary 41A21, 42C05.

1. The statement of the Riemann-Hilbert problem

Let \( \Delta_j = [c_{1,j}, c_{2,j}] \subset \mathbb{R}, \ j = 1, 2, \) be two intervals which are symmetric with respect to the origin. This means that \( c_{1,1} = -c_{2,2} \) and \( c_{1,2} = -c_{2,1}. \) For each \( j = 1, 2, \) we take a holomorphic function \( h_j, \) on a neighborhood \( \mathcal{V}_{h_j} \) of \( \Delta_j, \) i.e. \( h_j \in H(\mathcal{V}_{h_j}). \) Let us define the system of measures \((\sigma_1, \sigma_2)\) where \( \sigma_1 \) and \( \sigma_2 \) have the differential form

\[
d\sigma_j(x) = \frac{h_j(x)dx}{\sqrt{(x-c_{1,j})(c_{2,j}-x)}}, \quad x \in \Delta_j, \ j = 1, 2.
\]

This system \((\sigma_1, \sigma_2)\) belongs to the class of Angelesco systems introduced by Angelesco in [1]. Fix a multi-index \( n = (n_1, n_2), \) we say that a polynomial \( Q_n \neq 0 \) is a type II multiple-orthogonal polynomial corresponding to a system \((\sigma_1, \sigma_2), \) if \( \deg Q_n \leq |n| = n_1 + n_2 \) and \( Q_n \) satisfies the following orthogonality conditions

\[
\int_{\Delta_j} x^\nu Q_n(x)d\sigma_j(x) = 0, \quad \nu = 0, \ldots, n_j - 1, \ j = 1, 2.
\]

It is well known that for any multi-index \( n = (n_1, n_2), \) the polynomial \( Q_n \) has exactly \( n_1 + n_2 \) simple zeros lying in \( \overset{\circ}{\Delta_1} \cup \overset{\circ}{\Delta_2}, \) where \( \overset{\circ}{\Delta_j} \) denotes the interior set of \( \Delta_j, \ j = 1, 2. \) Our propose in the present article consists in obtaining results about the strong asymptotic development of sequences of multi-orthogonal

Received October 26, 2007.

The work of the first author was supported by CMUC/FCT, the work of the second author was supported by grant SFRH/BPD/31724/2006 from Fundação para a Ciência e a Tecnologia and the work of the third author was supported by UI Matemática e Aplicações from University of Aveiro.
polynomials $\{Q_n : n \in \mathbb{Z}^2\}$. An effective method for such study with this kind of “very well” measures, is analyzing of the Riemann-Hilbert problem for multi-orthogonal polynomials, which was introduced in [5]. Let us consider a $3 \times 3$ square matrix, $Y$, whose entries are complex functions $Y_{s,k} : \mathbb{C} \to \mathbb{C}$, $s,k = 1,2,3$. Given a point $x \in \Delta_1 \cup \Delta_2$, the following matricial limits, where $z \in \mathbb{C} \setminus (\Delta_1 \cup \Delta_2)$ tending to $x$, represent the formal pontual limits of all entries of $Y$ at the same time:

$$
\lim_{z \to x} Y(z) = Y_+(x), \quad \Im(m(z)) > 0
$$

$$
\lim_{z \to x} Y(z) = Y_-(x), \quad \Im(m(z)) < 0.
$$

Let $\delta_{s,k}$ denote the Kroneker delta function. Let us look for a matrix function $Y$ which satisfies the following conditions:

1. The entries of $Y$, $Y_{s,k}$, belongs to $H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$, which we write as $Y \in H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$;

2. For each $\Delta_j$, $j = 1,2$, the so called jump condition takes place

$$
Y_+(x) = Y_-(x) \begin{pmatrix}
1 & \frac{2\pi i \delta_{1,j} h_1(x) dx}{\sqrt{(x-c_{1,1})(c_{2,1}-x)}} & \frac{2\pi i \delta_{2,j} h_2(x) dx}{\sqrt{(x-c_{1,2})(c_{2,2}-x)}} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad x \in \overset{\circ}{\Delta}_j;
$$

3. Given a multi-index $n = (n_1, n_2)$, we require the following asymptotic condition at infinity,

$$
Y(z) \begin{pmatrix}
z^{-|n|} & 0 & 0 \\
0 & z^{n_1} & 0 \\
0 & 0 & z^{n_2}
\end{pmatrix} = \mathbb{I} + O\left(1/z\right) \quad \text{as} \quad z \to \infty,
$$

where $\mathbb{I}$ is the identity matrix with rank 3;

4. For each $i,j = 1,2$, we set the following behavior around the endpoints $c_{i,j}$,

$$
Y(z) = O\left(\begin{pmatrix}
1 & \frac{\delta_{1,j}}{\sqrt{|z-c_{i,j}|}} & \frac{\delta_{1,j}}{\sqrt{|z-c_{i,j}|}} \\
1 & \frac{\delta_{2,j}}{\sqrt{|z-c_{i,j}|}} & \frac{\delta_{2,j}}{\sqrt{|z-c_{i,j}|}} \\
1 & \frac{\delta_{1,j}}{\sqrt{|z-c_{i,j}|}} & \frac{\delta_{1,j}}{\sqrt{|z-c_{i,j}|}}
\end{pmatrix}\right).
$$

This problem, which consists in finding the matrix function $Y$, was called in [5] a Riemann-Hilbert problem for type II multiple orthogonal polynomials, and
for the system of measures \((\sigma_1, \sigma_2)\), RHP in short. The solution \(Y\) is unique and has the form

\[
Y(z) = \begin{pmatrix}
Q_n(z) & -\int_{\Delta_1} Q_n(x) \frac{d\sigma_1(x)}{z-x} & -\int_{\Delta_2} Q_n \frac{d\sigma_2(x)}{z-x} \\
d_1 Q_{n_1}(z) & -\int_{\Delta_1} Q_{n_1}(x) \frac{d\sigma_1(x)}{z-x} & -\int_{\Delta_2} Q_{n_1} \frac{d\sigma_2(x)}{z-x} \\
d_2 Q_{n_2}(z) & -\int_{\Delta_1} Q_{n_2}(x) \frac{d\sigma_1(x)}{z-x} & -\int_{\Delta_2} Q_{n_2} \frac{d\sigma_2(x)}{z-x}
\end{pmatrix}
\]

(1)

with \(d_i^{-1} = -\int_{\Delta_i} x^{n_i-1} Q_{n_i}(x) d\sigma_i(x), \ i = 1, 2\), and if \(n = (n_1, n_2)\), \(n_1 = (n_1 - 1, n_2)\) and \(n_2 = (n_1, n_2 - 1)\).

The key of our procedure is based in finding the relationship between \(Y\) and a matrix function \(R\) which is the solution of another RHP with the following formulation. Suppose that \(\gamma\) is a closed simple and smooth contour on the complex plane \(\mathbb{C}\), then find a matrix function, \(R\), such that:

1. \(R: \mathbb{C} \to \mathbb{C}^{3 \times 3}\) belongs to \(H(\mathbb{C} \setminus \gamma)\);
2. \(R_+ (\xi) = R_-(\xi) V_n(\xi), \ \xi \in \gamma\);
3. \(R(z) \to I\) as \(z \to \infty\),

where \(V_n\) is a \(3 \times 3\) matrix function, which is called the jump matrix.

Given an arbitrary \(3 \times 3\) matrix function \(K = [K_{s,k}]_{s,k}, \ s, k = 1, 2, 3\), defined on a open set \(\Omega \subset \mathbb{C}\) let us denote by \(\|K\|\) (respectively, \(\|K\|_\Omega\)) the matrix infinity norm which consists in the maximum sum of row’s entries modulus, defined for \(3 \times 3\) matrices, i.e.

\[
\|K\| = \max_{s=1,2,3} \sum_{k=1}^{3} |K_{s,k}|, \quad \text{(respectively, } \|K\|_\Omega = \sup_{\Omega} \|K\|).\]

**Theorem 1** (See Theorem 3.1 in [7]). Suppose that \(\Omega\) is an open set containing \(\gamma\). In condition (2) of the RHP for \(R\), let us require for \(V_n \in H(\Omega)\) that there exist constants \(C\) and \(\delta_n > 0\) for which

\[
\|V_n - I\|_\Omega < \delta_n.
\]

Then, any solution of the RHP for \(R\) satisfies that

\[
\|R(z) - I\| < C\|V_n - I\|_\Omega \quad \text{for every } z \in \mathbb{C} \setminus \gamma.
\]

Notice that if we know the relationship between \(R\) and \(Y\) and if we can also describe the development of \(R\) when \(|n| \to \infty\), we would have a description for the development of all entries of \(Y\) when \(|n| \to \infty\), particularly for \(Y_{1,1}(z) = Q_n(z)\).
The RHP for $Y$ is not normalized in the sense that the conditions (3) at infinity for $Y$ and $R$ are different. In order to normalize the RHP, we are going to modify $Y$ in such a way that we set another RHP with the same contours (possibly different jump conditions), for which the solution tends to the identity matrix as $z \to \infty$. For normalizing we need to take into account the behavior of $Y(z)$ for large $z$. This behavior depends on the distribution of the zeros of the multiple-orthogonal polynomials. The zero distribution of the orthogonal polynomials is usually given by an extremal problem in logarithmic potential theory. In section 2 we introduce some concepts and results which we will need about this theory and we will normalized the Riemann-Hilbert problem at infinity. In section 3 such a Riemann-Hilbert problem with oscillatory and exponentially decreasing jumps can be analyzed by using the steepest descent method introduced by Deift and Zhou (see [3, 4]).

2. The equilibrium problem and the normalization at infinity

Let us fix $j \in \{1, 2\}$. $\mathcal{M}_{1/2}(\Delta_j)$ denotes the set of all finite Borel measures whose supports, i.e. $\text{supp}(\cdot)$, are contained in $\Delta_j$ with total variation $1/2$. Take $\mu_j \in \mathcal{M}_{1/2}(\Delta_j)$ and define its logarithmic potential as follows

$$V^{\mu_j}(z) = \int \log \frac{1}{|z - x|} d\mu_j(x), \quad z \in \mathbb{C}.$$  

For each pair of measures $(\mu_1, \mu_2)$, where $\mu_j \in \mathcal{M}_{1/2}(\Delta_j)$, $j = 1, 2$, we define the quantities

$$m_j(\mu_1, \mu_2) = \min_{x \in \Delta_j} (2V^{\mu_j}(x) + V^{\mu_k}(x)), \quad j, k = 1, 2, \quad j \neq k.$$  

The following Proposition is deduced immediately from the results of [6].

**Proposition 1.** There exists a unique pair $(\bar{\mu}_1, \bar{\mu}_2) \in \mathcal{M}_{1/2}(\Delta_1) \times \mathcal{M}_{1/2}(\Delta_2)$, which satisfies for $j, k = 1, 2$

$$2V^{\bar{\mu}_j}(x) + V^{\bar{\mu}_k}(x) = m_j(\bar{\mu}_1, \bar{\mu}_2) = m_j, \quad x \in \text{supp}(\bar{\mu}_j) = \Delta_j, \quad j \neq k.$$  

For each $j = 1, 2$ the measure $\bar{\mu}_j$ has the following differential form

$$d\bar{\mu}_1(x) = \frac{\rho_1(x)dx}{\sqrt{(x - c_{1,1})(c_{2,1} - x)}}, \quad d\bar{\mu}_2(x) = \frac{\rho_2(x)dx}{\sqrt{(x - c_{1,2})(c_{2,2} - x)}},$$

where $\rho_j \in H(\mathcal{V}_{\rho_j})$, with $\mathcal{V}_{\rho_j}$ denoting an open set which contains $\Delta_j$. 
The pair \((\bar{\mu}_1, \bar{\mu}_2)\) is called extremal or equilibrium pair of measures with respect to \((\Delta_1, \Delta_2)\). Let us denote for each \(j = 1, 2\) the analytic potentials

\[
g_j(z) = \int_{\Delta_j^*} \log(z - x) d\bar{\mu}_j(x) = -V\bar{\mu}_j(z) + i \int_{\Delta_j^*} \arg(z - x) d\mu_j(x),
\]

where \(\Delta_j^*\) is the support of the extremal measure, \(\bar{\mu}_j\), that coincides in our case with \(\Delta_j\), for \(j = 1, 2\). Substituting the potential logarithmic in Proposition 1 we obtain for each \(j, k = 1, 2\) with \(j \neq k\) that

\[
-[g_{j+} + g_{j-}] (x) - g_{k-}(x) = m_j, \quad x \in \Delta_j.
\]

Observe that

\[
(g_{j+} - g_{j-}) (x) = \begin{cases} 
0 & \text{if } c_{2,j} \leq x \\
i\pi & \text{if } c_{1,j} \geq x \\
2i\pi \int_x^{c_{2,j}} d\bar{\mu}_j(t) & \text{if } x \in \Delta_j
\end{cases}.
\]

In what follows all the multi-indices will have the form \(n = (n, n)\). Let us introduce the matrices

\[
G(z) = \begin{pmatrix}
e^{-2n(g_1(z) + g_2(z))} & 0 & 0 \\
0 & e^{2ng_1(z)} & 0 \\
0 & 0 & e^{2ng_2(z)}
\end{pmatrix}, \quad L = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{-2nm_1} & 0 \\
0 & 0 & e^{-2nm_2}
\end{pmatrix}.
\] (2)

We define the matrix function \(T = LYGL^{-1}\), where \(L, G\) are as in (2) and \(Y\) is given by (1). Hence \(T\) is the unique solution of the RHP:

1. \(T \in H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))\);
2. \(T_+(x) = T_-(x)M(x), \; x \in \Delta_1 \cup \Delta_2\);
3. \(T(z) = \mathbb{I} + \mathcal{O}(1/z)\) as \(z \to \infty\);
4. \(T\) and \(Y\) have the same behavior on the endpoints of the intervals \(\Delta_j\), for \(j = 1, 2\),

where the jump matrix \(M\) has the form

\[
M(x) = \begin{pmatrix}
e^{-2nix \int_x^{c_{2,j}} d\bar{\mu}_j(t)} & 2\delta_{j,1}i\pi \omega_1(x) & 2\delta_{j,2}i\pi \omega_2(x) \\
0 & e^{2n\delta_{j,1}i\pi \int_x^{c_{2,1}} d\bar{\mu}_1(t)} & 0 \\
0 & 0 & e^{2n\delta_{j,2}i\pi \int_x^{c_{2,2}} d\bar{\mu}_2(t)}
\end{pmatrix}, \; x \in \Delta_j.
\]
3. The opening of the lens

For each \( j = 1, 2 \), let \( \phi_j \) denote the function defined by

\[
\phi_j(z) = i\pi \int_{z}^{c_{2,j}} d\mu_j(t) \quad \text{for} \quad z \in \mathcal{V}_j = \mathcal{V}_{\rho_j}(\Delta^*_j) \cap \mathcal{V}_{h_j}(\Delta_j).
\]

Notice that \( \phi_j(x) = i\pi \int_{x}^{c_{2,j}} d\mu_j(t) \) is purely imaginary and its derivative

\[
\phi'_j(x) = -i\pi \frac{\rho_j(x)}{\sqrt{(x - c_{1,j})(c_{2,j} - x)}},
\]

where \( -\pi \rho_j(x)/\sqrt{(x - c_{1,j})(c_{2,j} - x)} < 0, \quad x \in \Delta_j. \)

Rewrite \( \phi_j(z) = U_j(z) + iV_j(z) \in H(\mathcal{V}_j). \) By the Cauchy-Riemann conditions we have that the real part of \( \phi_j, \Re \phi_j, \) is an increasing function on any point \( z \in \mathcal{V}_j \) with \( \Im m(z) > 0. \) Since \( \Re \phi_j \) is zero in \( \Delta_j, \) it is positive in such point. Notice \( \phi_j(x) = -\phi_j(x), \quad x \in \Delta_j, \) hence we can proceed analogously when \( \Im m(z) < 0. \)

We analyze the jump function in \( \Delta_j, \) i.e.

\[
M(x) = \begin{pmatrix}
e^{-2n\phi_j(x)} & 2\pi i\delta_{j,1} w_1(x) & 2\pi i\delta_{j,2} w_2(x) \\
0 & e^{-2\delta_{j,1} n\phi_1(x)} & 0 \\
0 & 0 & e^{-2\delta_{j,2} n\phi_2(x)}
\end{pmatrix}
\times
\begin{pmatrix}
0 & 2\pi i\delta_{j,1} w_1(x) & 2\pi i\delta_{j,2} w_2(x) \\
-\frac{\delta_{j,1}}{2\pi i w_1(x)} & \delta_{j,2} & 0 \\
-\frac{\delta_{j,2}}{2\pi i w_2(x)} & 0 & \delta_{j,1}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
-\delta_{j,1} e^{-2n\phi_1(x)} & 1 & 0 \\
\frac{2\pi i w_1(x)}{-2\pi i w_2(x)} & 0 & 1
\end{pmatrix}.
\]

Now we are going to follow an analogous procedure as in section 9 of [7]. Let us fix a \( \delta > 0 \) such that the intervals \([c_{1,j} - \delta, c_{1,j}]\) and \([c_{2,j}, c_{2,j} + \delta]\) are subsets of \( \mathcal{V}_j, \quad j = 1, 2, \) and for each interval let us define two curves \( \Sigma_{j+}, \Sigma_{j-} \) in \( \mathcal{V}_j, \) which goes from \( c_{1,j} \) to \( c_{2,j}, \) where \( \Sigma_{j\pm} = [c_{1,j} - \delta, c_{1,j}] \cup [c_{2,j} + \delta, c_{2,j} + \delta], \) with the elements of the curves \( \Sigma_{j\pm}^* \) satisfying that if \( z \in \Sigma_{j\pm}^* \), then \( 0 < \pm \Im m(z) \) (cf. Figure 1). Set \( \Gamma_{j\pm} \) the domains that lie between \( \Sigma_{j\pm} \) and \( \Delta_j. \) Let us introduce the matrix function \( S, \) defined by

\[
S(z) = T \begin{pmatrix}
\frac{1}{\delta_{j,1} e^{-2n\phi_1(z)}} & 0 & 0 \\
\frac{2\pi i w_1(z)}{2\pi i w_2(z)} & 1 & 0 \\
\frac{\delta_{j,2} e^{-2n\phi_2(z)}}{2\pi i w_2(z)} & 0 & 1
\end{pmatrix}, \quad z \in \Gamma_{j\pm}, \quad \text{and} \quad S \equiv T, \quad \text{outside.} \quad (3)
\]
Hence on open intervals \(]c_{1,j} - \delta, c_{1,j}[^{\text{]}\text{]}}, \] \(]c_{2,j}, c_{2,j} + \delta[^{\text{]}\text{]}}, \] there are two combined jumps. For the function \(w_j\) we have that
\[
w_{j+}(x) = -w_{j-}(x), \quad x \in \mathbb{R} \cap \mathcal{V}(\Delta_j) \setminus \Delta_j, \quad j = 1, 2.
\]
Observe that
\[
\begin{pmatrix}
1 & 0 & 0 \\
\delta_{1,j} e^{-2n\phi_1} & 1 & 0 \\
\delta_{3,j} e^{-2n\phi_2} & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
\delta_{1,j} e^{-2n\phi_1} & 1 & 0 \\
\delta_{3,j} e^{-2n\phi_2} & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 \\
\frac{1}{w_{1-}} + \frac{1}{w_{1+}} & \delta_{1,j} e^{-2n\phi_1} & 1 & 0 \\
\frac{1}{w_{2-}} + \frac{1}{w_{2+}} & \delta_{3,j} e^{-2n\phi_2} & 0 & 1
\end{pmatrix} = I,
\]
which means that \(S\), defined by (3), is analytic function across \(]c_{1,j} - \delta, c_{1,j}[^{\text{]}\text{]}}, \) \(]c_{2,j}, c_{2,j} + \delta[^{\text{]}\text{]}}, \) \(j = 1, 2\). Let \(\gamma_j, j = 1, 2\), be closed contours with the clockwise direction, such that for each \(j = 1, 2\), \(\gamma_j = \Sigma_j^{*} \cup \Sigma_{j+}^{*}\). We have changed the direction of the curve \(\Sigma_{j+}^{*}\). The function \(S\) satisfies the RHP:

1. \(S \in H(\mathbb{C} \setminus \bigcup_{j=1,2}(\Delta_j \cup \gamma_j))\);
2. The jump conditions for \(j = 1, 2\) are,
\[
S_{+}(x) = S_{-}(x) \begin{pmatrix} 0 & 2\delta_{1,j} \pi i w_1(x) & 2\delta_{2,j} \pi i w_2(x) \\ \delta_{1,j} & \delta_{2,j} & 0 \\ \frac{2\pi i w_1(x)}{2\pi i w_2(x)} & 0 & \delta_{1,j} \end{pmatrix}
\] if \(x \in \Delta_j\),
\[
S_{+}(z) = S_{-}(z) \begin{pmatrix} 1 & 0 & 0 \\ \pm \frac{\delta_{1,j} e^{-2n\phi_1(z)}}{2\pi i w_1(z)} & 1 & 0 \\ \pm \frac{\delta_{3,j} e^{-2n\phi_2(z)}}{2\pi i w_2(z)} & 0 & 1 \end{pmatrix}
\] if \(z \in \gamma_j \cap \{\pm \Im z < 0\}\);
3. \(S(z) = I + O(1/z)\) as \(z \to \infty\);
4. The conditions for the endpoints are the same as for \(T\).

**Figure 1.** Opening of lens
Now, we consider the limiting problem, because for the matrix $S$ the jump matrix function on each $\gamma_j$ for $j = 1, 2$ tends to the identity matrix when $|n| \to \infty$. We look for the matrix function $N$ which satisfies the following RHP:

1. $N \in H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$;
2. The jump conditions in $\Delta_j$ for $j = 1, 2$ are,
   \[
   N_+(x) = N_-(x) \begin{pmatrix}
   0 & 2\delta_{1,j}\pi w_1(x) & 2\delta_{2,j}\pi w_2(x) \\
   -\frac{\delta_{1,j}}{2\pi i w_1(x)} & \delta_{2,j} & 0 \\
   -\frac{\delta_{2,j}}{2\pi i w_2(x)} & 0 & \delta_{1,j}
   \end{pmatrix};
   \tag{4}
   \]
3. $N(z) = I + \mathcal{O}(1/z)$ as $z \to \infty$;
4. $N$ satisfies the same conditions for the endpoints as $S$.

Set $N = KD$, where $D$ is a diagonal matrix function and $K = [K_{k,l}]_{k,l}$, $k, l = 1, 2$, is the solution of the RHP:

1. $K \in H(\mathbb{C} \setminus (\Delta_1 \cup \Delta_2))$;
2. The jump conditions in $\Delta_j$ for $j = 1, 2$ are, because of (4),
   \[
   K_+(x) = K_-(x) \begin{pmatrix}
   0 & 2\delta_{1,j}\pi i (\sqrt{(z-c_{2,j})(z-c_{1,j})}) & 2\delta_{2,j}\pi i (\sqrt{(z-c_{2,j})(z-c_{1,j})}) \\
   -\frac{\delta_{1,j}}{2\pi i (\sqrt{(z-c_{2,j})(z-c_{1,j})})} & \delta_{2,j} & 0 \\
   -\frac{\delta_{2,j}}{2\pi i (\sqrt{(z-c_{2,j})(z-c_{1,j})})} & 0 & \delta_{1,j}
   \end{pmatrix};
   \tag{5}
   \]
3. $K(z) = I + \mathcal{O}(1/z)$ as $z \to \infty$;
4. $K$ and $N$ have the same conditions for the endpoints.

Analogously to the ideas in [2], let us choose the branches of the square root which glue along the intervals $\Delta_j$, $j = 1, 2$, i.e.

\[
\left(\sqrt{(x-c_{2,j})(x-c_{1,j})}\right)_+ = -\left(\sqrt{(x-c_{2,j})(x-c_{1,j})}\right)_-, \quad x \in \Delta_j, \ j = 1, 2.
\]

For each $i = 1, 2, 3$, we rewrite (5) as

\[
\begin{cases}
(K_{i,2}(z))^\pm(x) = (K_{i,1}(z))^\mp(x), & x \in \Delta_j, \ j = 1, 2,
(K_{i,3})^+(x) = (K_{i,3})^-(x), & x \in \Delta_1
\end{cases}
\]
\[
\begin{aligned}
&\left\{ \left( \frac{\sqrt{(z-c_{2,2})(z-c_{1,2})}}{2\pi} K_{i,3}(z) \right) \pm (x) = (K_{i,1}(z))_\mp(x) \right. \quad x \in \Delta_2
\end{aligned}
\]

and we denote
\[
\psi^i_0(z) = K_{i,1}(z), \quad \text{and} \quad \psi^j_1(z) = \frac{\sqrt{(z-c_{2,j})(z-c_{1,j})}}{2\pi} K_{i,j+1}(z), \; j = 1, 2.
\]

Then from the relations (5), we may interpret each row \(i = 1, 2, 3\) of such matrix \(K\) as a function defined on a Riemann surface. Let \(\mathcal{R}\) define the Riemann surface which has two cuts. One of them connects the two branch points \(c_{1,1}\) and \(c_{2,1}\) with the cut in the interval \(\Delta_1 = [c_{1,1}, c_{2,1}]\). The other cut is made in the interval \(\Delta_2 = [c_{1,2}, c_{2,2}]\), to connect the two other branch points \(c_{1,2}\) and \(c_{2,2}\). The sheet \(\mathcal{R}_0\) is glued to another sheet \(\mathcal{R}_1\) along the cut \(\Delta_1\), and \(\mathcal{R}_0\) is also glued to \(\mathcal{R}_2\) along the interval \(\Delta_2\). Let us denote by \(\psi^i, \; i = 1, 2, 3\), three multi-valued function \(\psi^i = (\psi^i_0, \psi^i_1, \psi^i_2)\), such that its components \(\psi^i_l, \; i = 1, 2, 3, \; l = 0, 1, 2\), map the corresponding sheet \(\mathcal{R}_l\) on \(\mathbb{C}\), and satisfy for \(j = 1, 2\)
\[
\psi^i_0(x) = \psi^i_{j+}(x), \quad x \in \Delta_j, \quad \psi^i_j(z) = \mathcal{O}(1) \quad \text{as} \quad z \to c_{k,j}, \quad k = 1, 2; \quad (6)
\]

for \(j, k = 1, 2\), \(\psi^i_0(x) = \mathcal{O}(1)\), as \(z \to c_{k,j}\); around the infinity, and \(j = 1, 2\),
\[
\psi^i_0(z) = \delta_{i,1} + \mathcal{O}(1/z), \quad \psi^i_j(z) = z\delta_{i,j+1} + \mathcal{O}(1) \quad \text{as} \quad z \to \infty.
\]

The equalities (6) are equivalent to
\[
(\psi^i_0\psi^i_j)_+(x) = (\psi^i_0\psi^i_j)_-(x), \quad x \in \Delta_j, \quad j = 1, 2.
\]

From Liouville’s theorem, it is easy to see that \((\psi^i_0\psi^i_1\psi^i_2)(z) \equiv 1, \; z \in \overline{\mathbb{C}}\). This implies that \(\psi^i_l, \; l = 0, 1, 2, \; i = 1, 2, 3\), do not become zero. Hence
\[
(\psi^i_0\psi^i_k)(z) = \frac{1}{\psi^i_l(z)} \in H(\overline{\mathbb{C}} \setminus \Delta_l), \quad l, k = 1, 2, \quad l \neq k, \quad i = 1, 2, 3.
\]

We obtain that for each \(l = 1, 2\), that is \(x \in \Delta_l\),
\[
\psi^i_0(x)\psi^i_k(x) = \frac{1}{\psi^i_l(x)} \quad \text{or equivalently} \quad (\psi^i_+\psi^i_1)(x) = \frac{1}{\psi^i_k(x)}.
\]

That yields to the problems for \(\psi^i_l, \; l = 1, 2:\)
\[
\bullet \; \psi^i_l \in H(\overline{\mathbb{C}} \setminus \Delta_l);
\]
\[
(\psi_{l+}^i \psi_{l-}^i)(x) = 1/\psi_k^i(x), \quad x \in \hat{\Delta}_l; \\
\psi_l^i(z) = z\delta_{l,l+1} + \mathcal{O}(1) \text{ as } z \to \infty; \\
\psi_l^i(z) = \mathcal{O}(1) \text{ as } z \to c_{k,l}, \quad k = 1, 2.
\]

This problem is equivalent to the system of integral equations:

\[
\psi_0^i(z) = \frac{1}{\psi_1^i(z) \psi_2^i(z)} \\
\psi_1^i(z) = \exp \left( \frac{\sqrt{(z - c_{1,l})(z - c_{2,l})}}{2\pi} \int_{\Delta_l} \frac{\log \psi_k^i(x)}{\sqrt{(x - c_{1,l})(c_{2,l} - x)}} \frac{dx}{z - x} + \delta_{l+1}g_{\Delta_l}(z) \right),
\]

for \( l = 1, 2 \), with \( z \in \C \setminus \hat{\Delta}_l \), and \( g_{\Delta_j} \) is the analytic function which tends to \( \infty \) as \( \log z \), and whose real part vanishes in \( \Delta_j, \quad j = 1, 2 \).

Let us find the diagonal \( 3 \times 3 \) matrix function \( D \) with diagonal elements \( D_0, D_1, D_2 \), such that \( N(z) \equiv K(z)D(z) \). The conditions (5) yield that entries of \( D \) must satisfy the following conditions

\[
\begin{align*}
\{ & h_j(x)\psi_0^i(x) = D_{j+}^i(x) \\
& \psi_k(x) = \psi_{k+}(x) \}
\text{ when } x \in \hat{\Delta}_j, \quad j, k = 1, 2, \quad k \neq j.
\end{align*}
\]

Analogously to the function \( \psi_l^i \), we obtain the following problem for the entries of \( D \):

\[
\begin{align*}
i) & \quad (D_0D_1D_2) \equiv 1, \text{ which implies that for each } l = 0, 1, 2, \text{ } D_l \text{ does not become zero;} \\
ii) & \quad D_l \in H(\C \setminus \hat{\Delta}_l), \quad l = 1, 2; \\
iii) & \quad (D_{l+}D_{l-})(x) = h_s(x)/(D_s(x)), \quad s = 1, 2, \quad s \neq l, \quad x \in \hat{\Delta}_l, \quad l = 1, 2; \\
iv) & \quad D_l(z) = 1 + \mathcal{O}(1/z) \text{ as } z \to \infty, \quad l = 1, 2; \\
v) & \quad D_l(z) = \mathcal{O}(1) \text{ as } z \to c_{k,l}, \quad k = 1, 2, \quad l = 1, 2.
\end{align*}
\]

This problem is equivalent to the following system of integral equations where \( l = 1, 2 \),

\[
\begin{align*}
D_l(z) &= \exp \left( \frac{\sqrt{(z - c_{1,l})(z - c_{2,l})}}{2\pi} \int_{\Delta_l} \frac{\log (D_k(x)/h_k(x))}{\sqrt{(x - c_{1,l})(c_{2,l} - x)}} \frac{dx}{z - x} \right) \\
D_0(z) &= \frac{1}{D_1(z)D_2(z)} \quad \text{ with } z \in \C \setminus \hat{\Delta}_l.
\end{align*}
\]
Let us take the three multi-valued function \((D_0(z), D_1(z), D_2(z))\). Notice
that for each \(l = 0, 1, 2\), the function \(D_l\) is another function which maps the
sheet \(\mathcal{R}_l\) on \(\mathbb{C}\). In our case the components of functions \((D_0(z), D_1(z), D_2(z))\)
satisfy the conditions iv) and v) required for \(D_l, l = 0, 1, 2\). Finally the
matrix function \(N\) has the form

\[
N(z) = \begin{pmatrix}
(D_0\psi_0^1(z)) & (D_1\psi_1^1(z)) & (D_2\psi_2^1(z)) \\
(D_0\psi_0^2(z)) & (D_1\psi_1^2(z)) & (D_2\psi_2^2(z)) \\
(D_0\psi_0^3(z)) & (D_1\psi_1^3(z)) & (D_2\psi_2^3(z))
\end{pmatrix}.
\tag{7}
\]

We define \(R(z) = S(z)N^{-1}(z)\). Since \(S\) and \(N\) have the same jump across \(\Delta_j, j = 1, 2\), hence \(R_+(x) = R_-(x)\) for \(x \in \Delta_j, j = 1, 2\). From the definition of \(R\),
and the endpoint conditions for \(N\), we can also deduce that \(c_{i,k}, i, k = 1, 2\),
are a removable singularity. Hence \(R\) is an analytic function across the full
intervals \(\Delta_1\) and \(\Delta_2\), and it has jumps on the curve \(\gamma\). Then we have the
following RHP for \(R\):

1. \(R \in H(\mathbb{C} \setminus (\gamma_1 \cup \gamma_2))\);
2. The jump conditions are for \(j = 1, 2\)

\[
R_+(z) = R_-(z) \begin{pmatrix}
1 & 0 & 0 \\
\pm \frac{\delta_{j,1}e^{-2n_1\phi_1(z)}}{2\pi n_1\nu_1(z)} & 1 & 0 \\
\pm \frac{\delta_{j,2}e^{-2n_2\phi_2(z)}}{2\pi n_2\nu_2(z)} & 0 & 1
\end{pmatrix}
\]

if \(z \in \gamma_j \cap \{\pm \Im z < 0\}\);

3. \(R(z) = I + \mathcal{O}(1/z)\).

Observe that matrix function \(R\) satisfies the hypothesis of the Theorem 1.
Then uniformly for \(z \in \mathbb{C} \setminus (\gamma_1 \cup \gamma_2)\), we have that \(R(z) = I + \mathcal{O}(e^{c|n|})\),
with \(c > 0\) and

\[
Y(z) = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{n|m_1|} & 0 \\
0 & 0 & e^{n|m_2|}
\end{pmatrix} (I + \mathcal{O}(e^{-c|n|}))
\]

\[
\times N(z) \begin{pmatrix}
e^{n|g_1+g_2|}(z) & 0 & 0 \\
0 & e^{-n|g_1(z)|} & 0 \\
0 & 0 & e^{-n|m_1+g_2(z)|}
\end{pmatrix},
\]

where \(N\) is given by (7).

Finally, we state the main result of this paper.
Theorem 2.

\[ Y_{1,1}(z) = Q_n(z) = D_0(z)\psi_0(z) e^{\|g_1+g_2\|_n}(1 + \mathcal{O}(e^{-c|n|})) , \]
as \( |n| \to \infty \).

References


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