Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 07–30

THE CARDINALITY OF ENDOMORPHISMS OF SOME ORIENTED PATHS: AN ALGORITHM

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Abstract: An endomorphism of a (oriented) graph is a mapping on the vertex set preserving (arcs) edges. In this paper we provide an algorithm to determine the cardinalities of endomorphism monoids of some (finite) directed paths, based on results on simple paths.

Keywords: simple path; oriented path; graph endomorphism; square lattice.
AMS Subject Classification (2000): 00A22, 08B35, 11B65.

1. Introduction

Our aim is the computation of the cardinalities of the endomorphism monoids of some finite oriented paths, providing some close formulas. We consider endomorphisms to be arc preserving mappings of the vertex set of a digraph into itself. Other types of endomorphisms were already considered elsewhere (cf., e.g., [3]). In [1], Arworn studied an algorithm to determine the cardinalities of endomorphism monoids of finite undirected paths. We recover some of that results giving new proofs. Similar but slightly different methods for counting paths in square lattices but only from $(0,0)$ to (i,i) are presented in [5]. We point out that undirected paths as very special trees have exactly two automorphisms and no other quasi-strong or strong endomorphisms. Moreover all endomorphisms are half strong which means that every edge has at least one preimage. So only the set of locally strong endomorphisms (i.e., endomorphisms such that for images which form an edge, every preimage of one end point of this edge has a neighbor in the preimage set of the other end point of this edge) is different from the group of automorphisms and the monoid of all endomorphisms if the path has at least 5 vertices (cf. [4]), but does not form a monoid if the path has 7 vertices or more, as can be seen by calculation.

Received October 29, 2007.
This work was supported by Chiang Mai University, CMUC - Centro de Matemática da Universidade de Coimbra, Srinakharinwirot University, the Thailand Research Fund and Commission on Higher Education, Thailand MRG5080075.

2. Square lattices

Consider the following complete square lattice (here with $i = 7$ and $j = 5$):

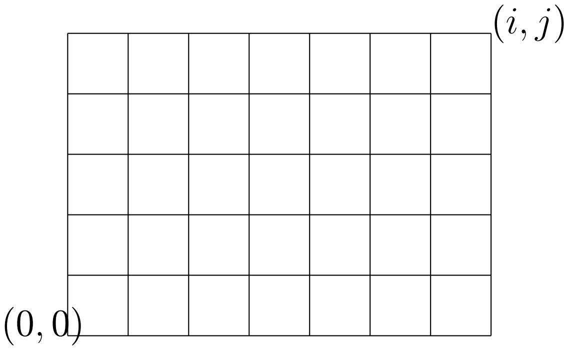

Figure 1

Notice that all shortest paths on this square lattice from the point $(0, 0)$ to the point (i, j) can be obtained going from the point $(0, 0)$ to the right or up only until arrival at the point (i, j). If we denote any step to the right by 1 and any step up by 0, then any such path will have i occurrences of 1 and j occurrences of 0. Conversely, any permutation of i copies of 1 and j copies of 0 will be a shortest path from the point $(0, 0)$ to the point (i, j). This fact is summarized in the next proposition which is essential in all paper and it is well known. We include a different proof, with the usual convention that $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ for nonnegative integers n, m such that $n \geq m$.

Theorem 2.1. *The number $M(i, j)$ of the shortest paths from the point $(0, 0)$ to the point (i, j) is*

$$M(i,j) = \binom{i+j}{j} .$$

Proof: We will use induction on j. For $j = 1$, i.e.,

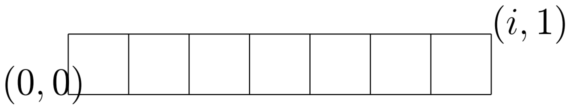

Figure 2

it is clear that $M(i, 1) = i + 1 = \binom{i+1}{1}$. Suppose now that the result is true for $j - 1$, and for any i, i.e., $M(i, j - 1) = \binom{i+j-1}{j-1}$. Then

$$M(i,j) = \sum_{k=0}^{i} M(k, j-1) = \sum_{k=0}^{i} \binom{k+j-1}{j-1} = \binom{i+j}{j} . \qquad \blacksquare$$

Naturally, we have $M(i,j) = M(j,i)$.

Another kind of square lattices is the so-called the *r-ladders* as it shown in Figure 3, for $r = 2$, $i = 7$, and $j = 5$:

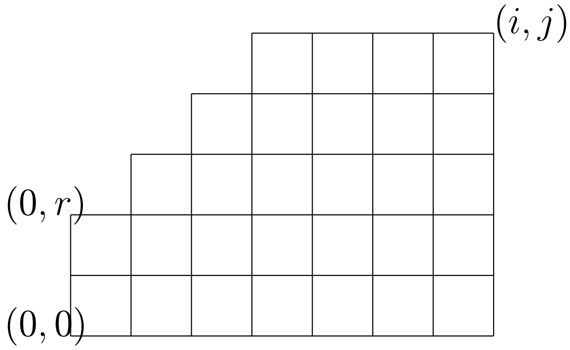

Figure 3

Let $M_r(i,j)$ denote the number of the shortest paths from the point $(0,0)$ to the point (i,j) in an r-ladder square lattice. If $j = r+1$, then deleting from $M(i,r+1)$ the only path which crosses the point $(0,r+1)$ we get $M_r(i,r+1)$. Supposing that

$$M_r(i,r+m) = \binom{i+r+m}{r+m} - \binom{i+r+m}{m-1},$$

for any $i \geq m$, we have

$$\begin{aligned} M_r(i,r+m+1) &= \sum_{\ell=1}^{i-m} M_r(m+\ell, r+m) \\ &= \sum_{\ell=1}^{i-m} \left[\binom{r+2m+\ell}{r+m} - \binom{r+2m+\ell}{m} \right] \\ &= \binom{i+r+m+1}{r+m+1} - \binom{i+r+m+1}{m}. \end{aligned}$$

We have just shown:

Theorem 2.2. *Let r and j be integers such that $1 \leq r+1 \leq j$. Then on the r-ladder square lattice, we have*

$$M_r(i,j) = \binom{i+j}{j} - \binom{i+j}{j-r-1},$$

i.e.,

$$M_r(i,j) = M(i,j) - M(i+r+1, j-r-1).$$

Notice that from Theorem 2.2 we have in particular, for example,

$$M_0(i,j) = M(i,j) - M(i+1,j-1)\ , \tag{2.1}$$

and

$$M_1(i,j) = M(i,j) - M(i+2,j-2)\ . \tag{2.2}$$

Consider now a *back r-ladder* square lattice as it is shown in Figure 4, for $r = 2$, $i = 7$ and $j = 5$:

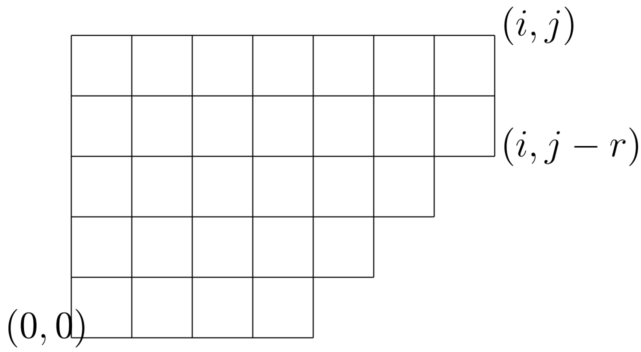

Figure 4

This case can be treated in the same way as the r-ladder square lattice before. In fact, let $m = i + j$. If $a_1, a_2, \ldots, a_m$ is a permutation of i copies of 1 and j copies of 0 which represents a shortest path from the point $(0,0)$ to the point (i,j) on the r-ladder square lattice, then the permutation $a_m, a_{m-1}, \ldots, a_1$ will be a permutation of a shortest path from the point $(0,0)$ to the point (i,j) on the back r-ladder square lattice. And *vice-versa*. Therefore the number of the shortest paths from the point $(0,0)$ to the point (i,j) on the back r-ladder square lattice is $M_r(i,j)$, the number of the shortest path on the r-ladder square lattice. This argument becomes obvious if we turn the back r-ladder square lattice 180 degrees, then it becomes an r-ladder square lattice.

3. Endomorphisms of simple paths

An *endomorphism* of a graph is a mapping on the vertex set preserving edges. In this section we will study the cardinalities of endomorphisms of a undirected path.

Let P_m be an undirected (or simple) path with length $m - 1$, where $m \geq 2$. We assume the vertex set as $\{0, 1, 2, \ldots, m-1\}$ with edge set

$\{\{i, i+1\}\ :\ \text{for } i = 0, \ldots, m-2\}$. We will denote the number of endomorphisms of the path P_m by $|\mathrm{End}(P_m)|$ and the number of endomorphisms of the path P_m which map 0 to r by $\varepsilon(P_m, r)$, for $r = 0, 1, \ldots, m-1$.

We will start giving an example of the procedure that will be used throughout. Let us determine the number $\varepsilon(P_4, 0)$ using a decision tree as follows:

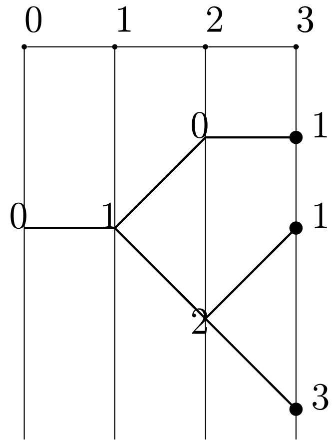

Figure 5

Figure 5 indicates under every point its possible images, assuming that 0 is fixed. This implies that 1 is fixed and that 2 can be mapped to 0 or stay fixed, and so on. From the decision tree we get $\varepsilon(P_4, 0) = 3$.

In order to find a general procedure for the computation of $\varepsilon(P_m, r)$, let us draw again Figure 5 identifying "leaves" which carry the same number as shown in Figure 6:

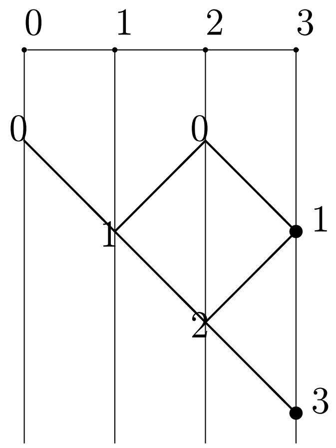

Figure 6

Therefore, the tree from Figure 5 becomes the ladder:

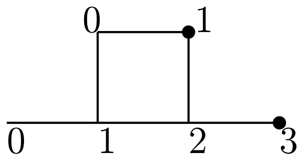

Figure 7

So we get

$$\begin{aligned}\varepsilon(P_4,0) &= M(3,0)+M_0(2,1)\\ &= \binom{3}{0}+\binom{3}{1}-\binom{3}{0}\\ &= 3\end{aligned}$$

We point out that we get the same square lattice in the determination $\varepsilon(P_4,3)$. In fact, we have in general:

Proposition 3.1. *Let m and r be integers such that $0 \le r \le m-1$. Then*

$$\varepsilon(P_m,r)=\varepsilon(P_m,m-r-1).$$

As an immediate consequence of the definition of endomorphism we have the next corollary:

Corollary 3.2. *Let m an integer, with $m \ge 2$. Then*

$$|End(P_m)| = \begin{cases} 2\displaystyle\sum_{\ell=0}^{\frac{m-3}{2}} \varepsilon(P_m,\ell)+\varepsilon(P_m,(m-1)/2) & \textit{if } m \textit{ is odd}\\ 2\displaystyle\sum_{\ell=0}^{\frac{m-2}{2}} \varepsilon(P_m,\ell) & \textit{if } m \textit{ is even.}\end{cases}$$

We compute now $\varepsilon(P_4,1)$:

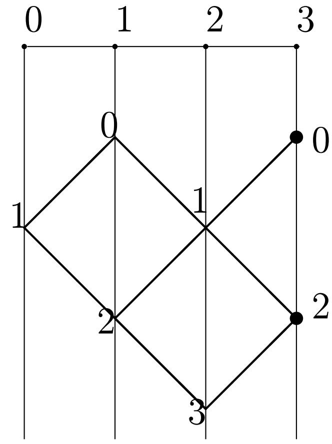

Figure 8

Therefore, we get the square lattice:

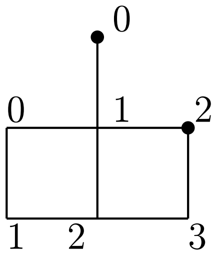

Figure 9

and, therefore,

$$\varepsilon(P_4, 1) = M(2, 1) + M(1, 1) = \binom{2+1}{1} + \binom{1+1}{1} = 5.$$

Finally, from Corollary 3.2 we get

$$|\mathrm{End}(P_4)| = 2(\varepsilon(P_4, 0) + \varepsilon(P_4, 1)) = 2(3 + 5) = 16\ .$$

4. General formulas for $\varepsilon(P_m, r)$

We will establish some close formulas for $\varepsilon(P_m, r)$, the number of endomorphisms on path P_m which fix r.

Lemma 4.1.

$$\varepsilon(P_m, 0) = \binom{m-1}{\lceil m/2 \rceil - 1} = \begin{cases} M(n, n-1) & \textit{if } m = 2n \\ M(n, n) & \textit{if } m = 2n+1. \end{cases}$$

Proof: If m is even, say $m = 2n$, we have the square lattice just below (here it was drawn for $n = 5$):

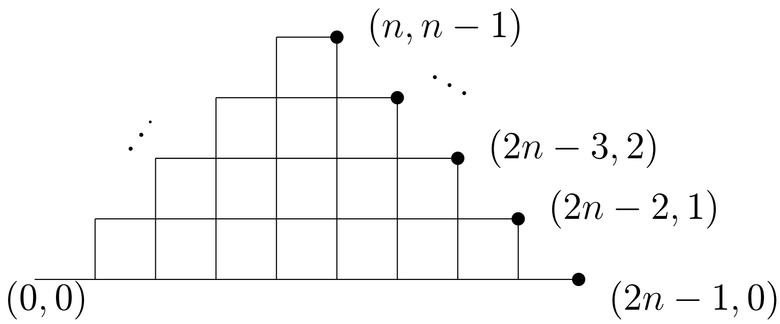

Figure 10

If $m = 2n+1$, we have the square lattice (drawn also for $n = 5$):

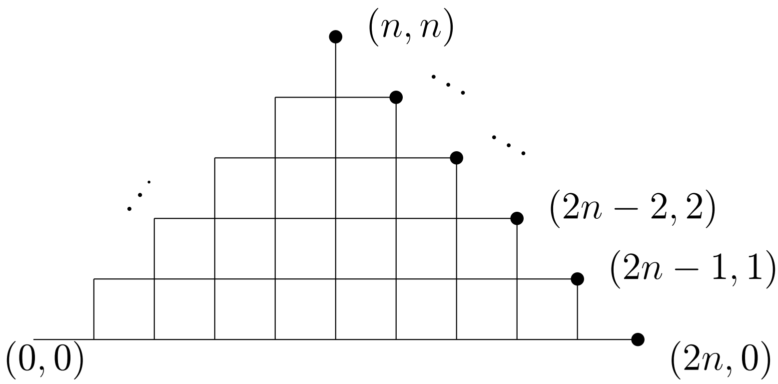

Figure 11

Hence, we have

$$\varepsilon(P_m, 0) = \sum_{\ell=0}^{\lceil \frac{m}{2} \rceil - 1} M_0(m-1-\ell, \ell)$$

and, from (2.1),

$$\varepsilon(P_m, 0) = \begin{cases} M(n, n-1) & \text{if } m = 2n \\ M(n, n) & \text{if } m = 2n+1. \end{cases}$$

■

Notice that $\varepsilon(P_{2n+1}, 0) = 2\varepsilon(P_{2n}, 0)$.

Lemma 4.2.

$$\varepsilon(P_{2n}, 1) = \binom{2n-1}{n-1} + \binom{2n-1}{n} - 1.$$

Proof: We have the ladder:

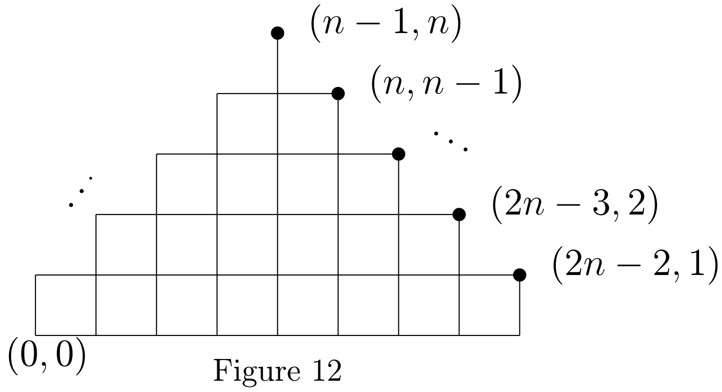

Figure 12

Then, from (2.2),

$$\begin{aligned}\varepsilon(P_{2n},1) &= M(2n-2,1)+M_1(2n-3,2)+M_1(2n-4,3)+\cdots \\ &\qquad +M_1(n,n-1)+M_1(n-1,n) \\ &= M(n,n-1)+M(n-1,n)-1.\end{aligned}$$

■

Lemma 4.3.

$$\varepsilon(P_m,2)=\binom{m-1}{\lceil\frac{m}{2}\rceil-2}+\binom{m-1}{\lceil\frac{m}{2}\rceil-1}+\binom{m-1}{\lceil\frac{m}{2}\rceil}-2.$$

Proof: Let us consider the following ladder for $m=2n$:

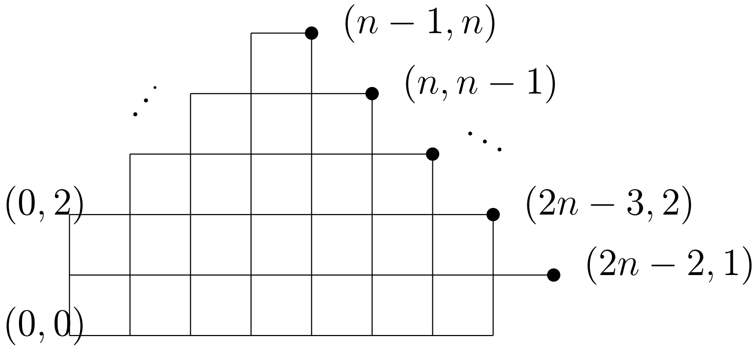

Figure 13

Then

$$\begin{aligned}\varepsilon(P_{2n},2) &= M_0(2n-2,1)+M(2n-3,2)+M_2(2n-4,3)+\cdots \\ &\qquad +M_2(n,n-1)+M_2(n-1,n) \\ &= M(n+1,n-2)+M(n,n-1)+M(n-1,n)-2\ .\end{aligned}$$

If $m=2n+1$, then

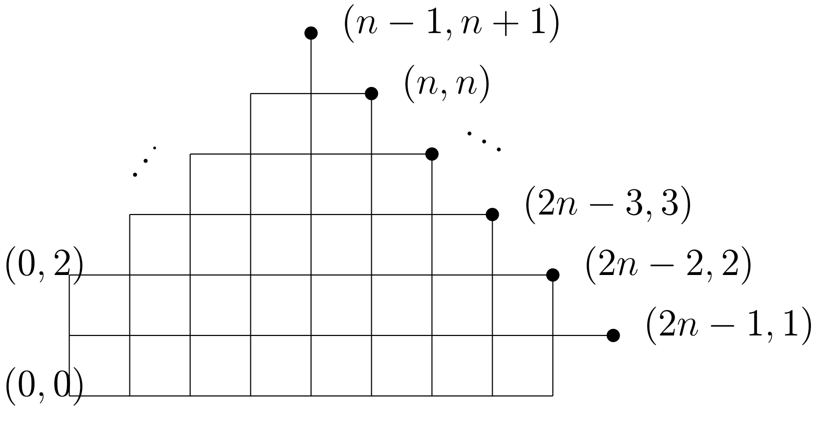

Figure 14

and

$$\begin{aligned}\varepsilon(P_{2n+1}, 2) &= M_0(2n-1, 1) + M(2n-2, 2) + M_2(2n-3, 3) + \cdots \\ &\qquad + M_2(n, n) + M_2(n-1, n+1) \\ &= M(n+1, n-1) + M(n, n) + M(n-1, n+1) - 2\ . \qquad \blacksquare\end{aligned}$$

Finally, we give general formulas for $\varepsilon(P_{2n}, 2t)$, $\varepsilon(P_{2n}, 2t-1)$, $\varepsilon(P_{2n+1}, 2t)$ and $\varepsilon(P_{2n+1}, 2t-1)$. The laders were drawn for $n = 6$ and $t = 2$ or $t = 3$.

Theorem 4.4. *For $1 \leq t \leq n-2$, we have*

$$\varepsilon(P_{2n}, 2t) = \sum_{\ell=n-t-1}^{n+t-1} \binom{2n-1}{\ell} - 2\sum_{\ell=0}^{t-1} \binom{2n-1}{\ell}\,;$$

and, for $2 \leq t \leq n-1$, we have

$$\varepsilon(P_{2n}, 2t-1) = \sum_{\ell=n-t}^{n+t-1} \binom{2n-1}{\ell} - 2\sum_{\ell=0}^{t-2} \binom{2n-1}{\ell} - \binom{2n-1}{t-1}.$$

Proof: To find $\varepsilon(P_{2n}, 2t)$, we consider the following lader:

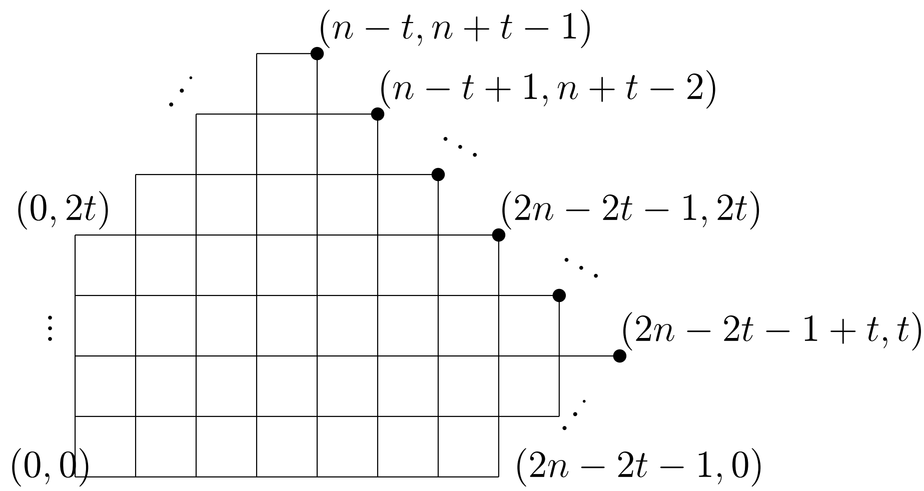

Figure 15

Then, for $1 \leq t \leq n-2$,

$$\begin{aligned}
\varepsilon(P_{2n}, 2t) &= M_0(2n-t-1, t) + M_2(2n-t-2, t+1) + M_4(2n-t-3, t+2) + \cdots \\
&\quad + M_{2(t-1)}(2n-2t, 2t-1) + M(2n-2t-1, 2t) + M_{2t}(2n-2t-2, 2t+1) \\
&\quad + \cdots + M_{2t}(n-t-1, n+t-2) + M_{2t}(n-t, n+t-1) \\
&= \sum_{\ell=n-t-1}^{n+t-1} M(2n-\ell-1, \ell) - 2\sum_{\ell=0}^{t-1} M(2n-\ell-1, \ell)\,.
\end{aligned}$$

Analogously, for $\varepsilon(P_{2n}, 2t-1)$, we have

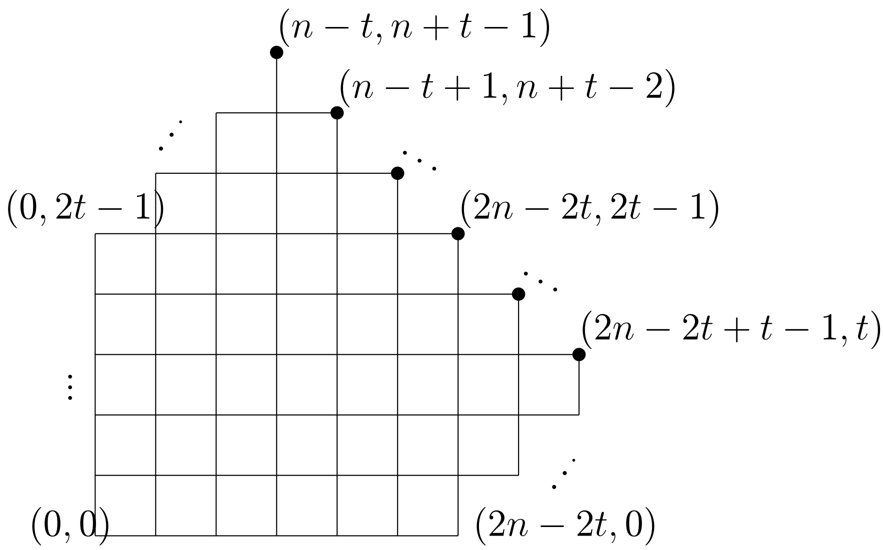

Figure 16

and

$$\begin{aligned}(P_{2n}, 2t-1) &= M_1(2n-t-1,t) + M_3(2n-t-2,t+1) + \cdots \\ &\quad + M_{2(t-2)+1}(2n-2t+1, t+(t-2)) + M(2n-2t, 2t-1) \\ &\quad + M_{2t-1}(2n-2t-1, 2t) + M_{2t-1}(2n-2t-2, 2t+1) + \cdots \\ &\quad + M_{2t-1}(n-t, n+t-1) \\ &= \sum_{\ell=n-t}^{n+t-1} M(2n-\ell-1,\ell) - 2\sum_{\ell=0}^{t-2} M(2n-\ell-1,\ell) - M(2n-t,t-1)\,.\end{aligned}$$

■

Theorem 4.5. *For* $1 \leq t \leq n-1$,

$$\varepsilon(P_{2n+1}, 2t) = \sum_{\ell=n-t}^{n+t} \binom{2n}{\ell} - 2\sum_{\ell=0}^{t-1} \binom{2n}{\ell},$$

and

$$\varepsilon(P_{2n+1}, 2t-1) = \sum_{\ell=n-t}^{n+t-1} \binom{2n}{\ell} - 2\sum_{\ell=0}^{t-2} \binom{2n}{\ell} - \binom{2n}{t-1}.$$

Proof: For the first case, we have:

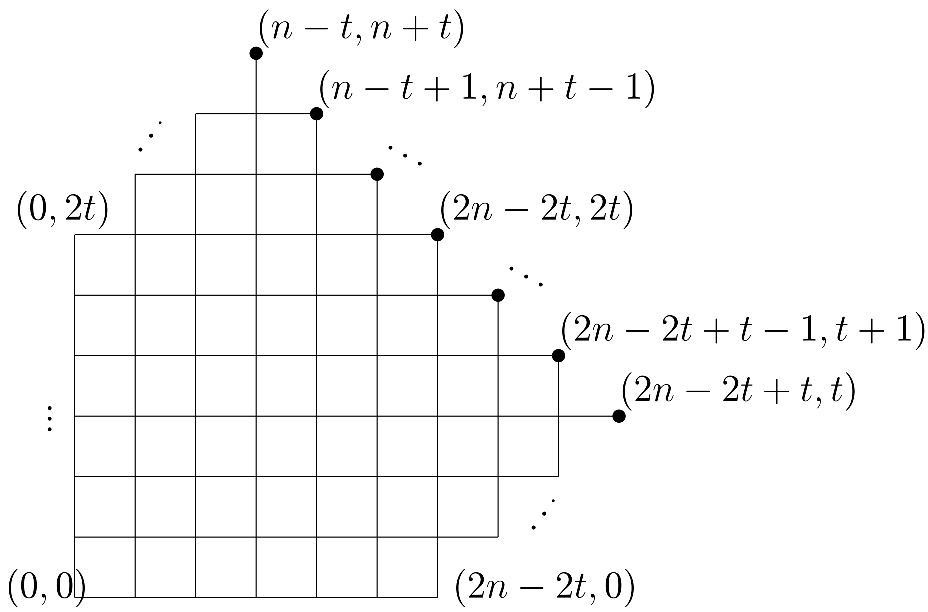

Figure 17

and therefore

$$\begin{aligned}\varepsilon(P_{2n+1},2t) &= M_0(2n-t,t)+M_2(2n-t-1,t+1)+\cdots+M_{2(t-1)}(2n-2t+1,t+(t-1))\\ &\quad +M(2n-2t,2t)+M_{2t}(2n-2t-1,2t+1)+M_{2t}(2n-2t-2,2t+2)+\cdots\\ &\quad +M_{2t}(n-t,n+t)\\ &= \sum_{\ell=n-t}^{n+t} M(2n-\ell,\ell)-2\sum_{\ell=0}^{t-1} M(2n-\ell,\ell)\,.\end{aligned}$$

On the other hand,

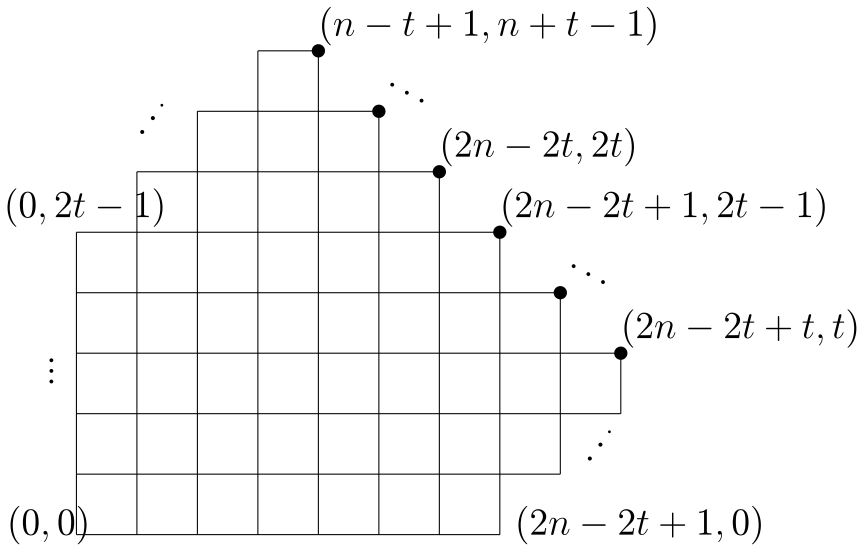

Figure 18

Then

$$\begin{aligned}\varepsilon(P_{2n+1},2t-1) &= M_1(2n-t,t)+M_3(2n-t-1,t+1)+\cdots+\\ &\quad +M_{2(t-2)+1}(2n-2t+1,t+(t-2))+M(2n-2t+1,2t-1)+\\ &\quad +M_{2t-1}(2n-2t,2t)M_{2t-1}(2n-2t-1,2t+1)+\\ &\quad +M_{2t-1}(n-t+2,n+t-2)+M_{2t-1}(n-t+1,n+t-1)\\ &= \sum_{\ell=n-t}^{n+t-1} M(2n-\ell,\ell)-2\sum_{\ell=0}^{t-2} M(2n-\ell,\ell)-M(2n-t+1,t-1)\,.\end{aligned}$$

■

5. Endomorphisms on complete folding oriented paths

A *complete folding* $\vec{P}_{m;k}$, with $m=tk+1$ for some positive integer t, is the oriented path with m vertices:

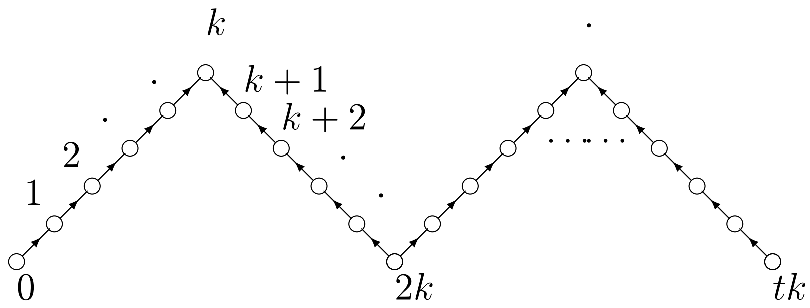

Figure 19

As before, we will denote the number of endomorphisms on complete folding $\vec{P}_{m;k}$ by $|\mathrm{End}(\vec{P}_{m;k})|$ and the number of endomorphisms on $\vec{P}_{m;k}$ which map 0 to r by $\varepsilon(\vec{P}_{m;k}, r)$.

Lemma 5.1. *Given two sets*

$$S = \{2tk \in V(\vec{P}_{m;k}) \ : \ t \textit{ is a nonnegative integer}\}$$

and

$$T = \{2tk + 1 \in V(\vec{P}_{m;k}) \ : \ t \textit{ is a nonnegative integer}\},$$

$f(S) \subset S$ and $f(T) \subset T$ for each endomorphism f on $\vec{P}_{m;k}$.

Since

$$|\mathrm{End}(\vec{P}_{m;k})| = |\mathrm{End}(\vec{P}_{t+1;1})| \ ,$$

we will be concentrated on $\vec{P}_{m;1}$ which from now on we will be simply denoted by $\vec{P}_m$.

Lemma 5.2. *Each endomorphism on complete folding oriented path $\vec{P}_m$, with $m \geq 3$ sends even vertices to even vertices and sends odd vertices to odd vertices.*

Analogously to Proposition 3.1, we may state:

Proposition 5.3. *If m is an odd integer and t be such that $0 \leq 2t \leq m - 1$, then $\varepsilon(\vec{P}_m, 2t) = \varepsilon(\vec{P}_m, m - 2t - 1)$.*

And similarly to Corollary 3.2,

Corollary 5.4. *Let n an integer, with $m \geq 1$. Then*

$$|End(\vec{P}_{2n})| = \sum_{\ell=0}^{n-1} \varepsilon(\vec{P}_{2n}, 2\ell) \quad \textit{and} \quad |End(\vec{P}_{2n+1})| = \sum_{\ell=0}^{n} \varepsilon(\vec{P}_{2n+1}, 2\ell) \ .$$

We will start giving an example of the procedure that will be used in this case. Let us determine the number $\varepsilon(\vec{P}_4, r)$ using a decision tree as follows:

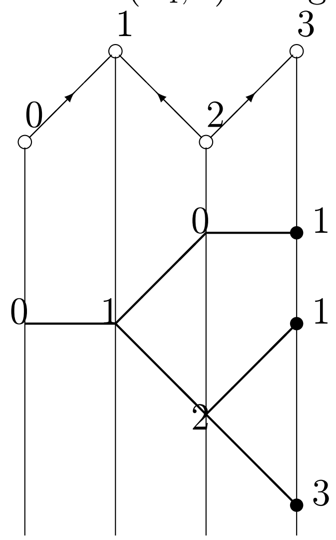

Figure 20

In Figure 20 it can be seen the possible images of every vertex, assuming that 0 is fixed. This implies that 1 is fixed and that 2 can be mapped to 0 or stay fixed etc. From the decision tree we have $\varepsilon(\vec{P}_4, 0) = 3$.

Notice that the decision tree is the same as the one arose in Figure 5. Therefore we have in this case the ladder of Figures 6 and 7, and $\varepsilon(\vec{P}_4, 0) = \varepsilon(P_4, 1) = \varepsilon(P_4, 0)$.

If we want to compute $\varepsilon(\vec{P}_4, 2)$ we get:

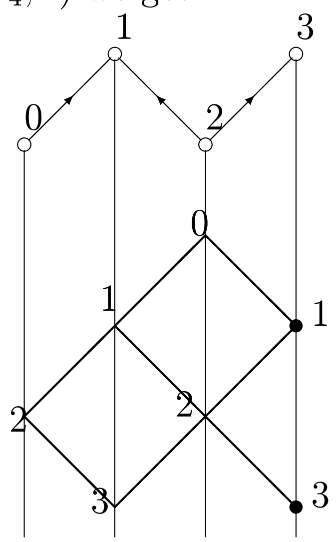

Figure 21

Comparing Figures 21 with Figures 8 and 9 we conclude $\varepsilon(\vec{P}_4, 2) = \varepsilon(P_4, 2)$. In fact we have in general:

Theorem 5.5. *For any integer t with $0 \leq t \leq m/2$,*

$$\varepsilon(\vec{P}_m, 2t) = \varepsilon(P_m, 2t)\ .$$

Finally we provide a table with $|\mathrm{End}(\vec{P}_m)|$ for $m \leq 20$.

m	1	2	3	4	5	6	7	8	9	10
$\lvert\mathrm{End}(\vec{P}_m)\rvert$	1	1	4	8	24	52	136	296	720	1556

m	11	12	13	14	15	16	17	18	19	20
$\lvert\mathrm{End}(\vec{P}_m)\rvert$	3624	7768	17584	37416	83024	175568	383904	807604	1746280	3657464

6. Endomorphisms on incomplete folding oriented paths

We consider now a more general kind of oriented paths. Let us start with the following oriented path:

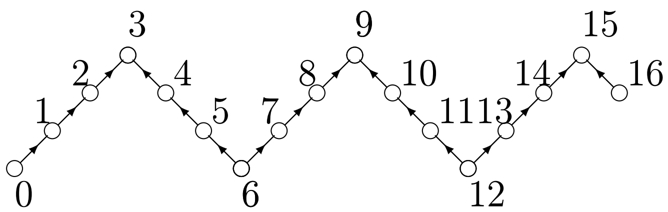

Figure 22

This path will be called the *incomplete* folding oriented path $\vec{P}_{17;3}$.

In general, we will focus our attention to $\varepsilon(\vec{P}_{m;k}, t)$, where

$$m - 1 = nk + t \quad \text{and} \quad 1 \leq t \leq k - 1\ . \tag{6.1}$$

$m - 1 = nk + t$ and $1 \leq t \leq k - 1$. The study will be depend on the parity of n. Some particular cases motivate the general case.

We proceed with the computation of $\varepsilon(\vec{P}_{17;3}, 0)$, $\varepsilon(\vec{P}_{17;3}, 6)$, $\varepsilon(\vec{P}_{21;3}, 0)$ and $\varepsilon(\vec{P}_{21;3}, 6)$, using square lattices as before. Let us start with $\varepsilon(\vec{P}_{17;3}, 0)$:

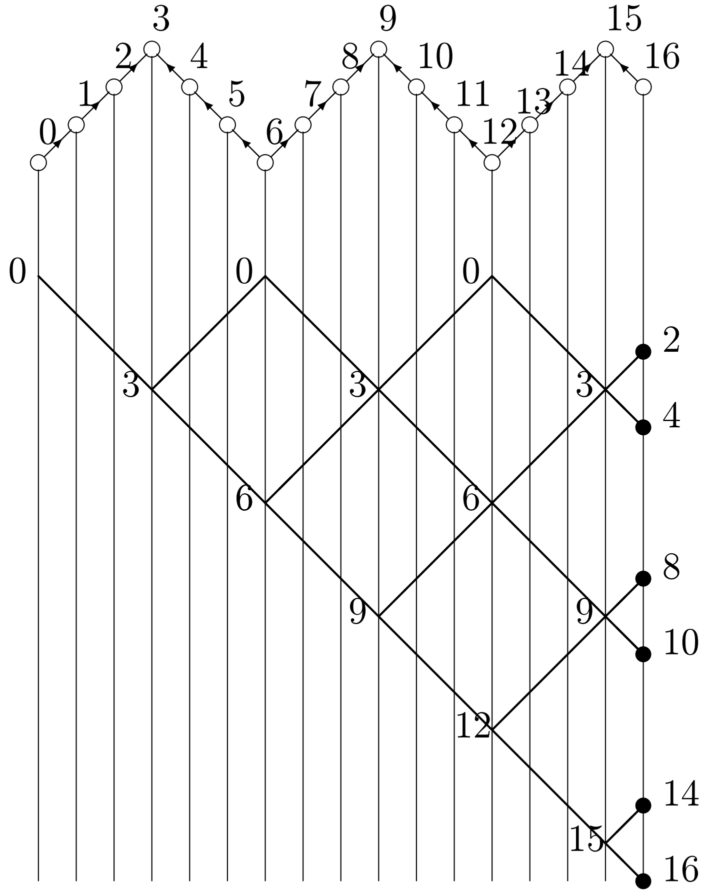

Figure 23

We get the ladder:

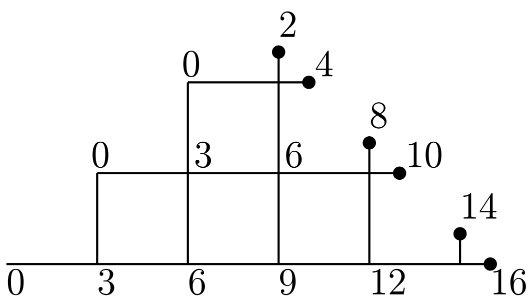

Figure 24

This is the same ladder achieve in Figure 10 for $n = 3$. Therefore $\varepsilon(\vec{P}_{17;3}, 0) = 2\,\varepsilon(P_6, 0)$.

Let us compute now $\varepsilon(\vec{P}_{17;3}, 6)$. We have

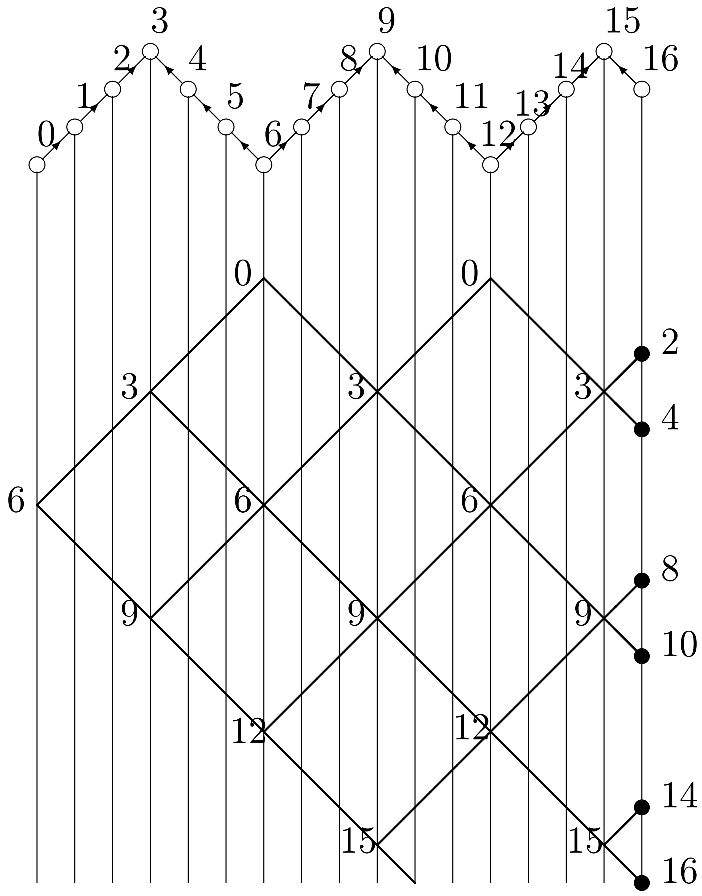

Figure 25

and

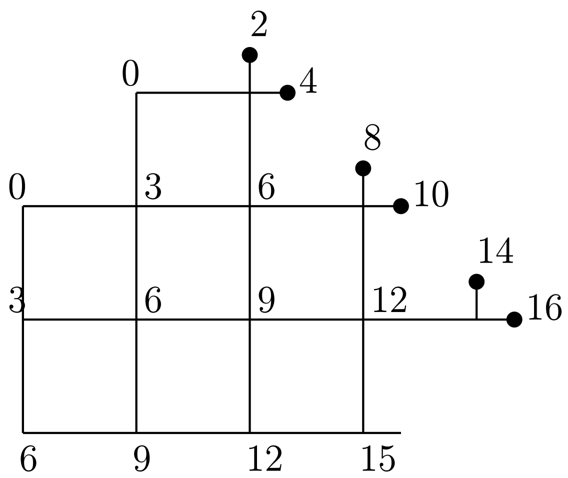

Figure 26

Hence, from Figure 13, we can conclude

$$\varepsilon(\vec{P}_{17;3}, 6) = 2\,\varepsilon(P_6, 2)\ .$$

Similarly,

$$\varepsilon(\vec{P}_{17;3}, 12) = 2\,\varepsilon(P_6, 4)(= 2\,\varepsilon(P_6, 1))\ .$$

Hence we can establish the general formula when n is odd in (6.1):

$$\varepsilon(\vec{P}_{m;k}, t) = 2\,\varepsilon\left(P_{n+1}, \frac{t}{k}\right)\ .$$

Next we compute $\varepsilon(\vec{P}_{21;3}, 0)$.

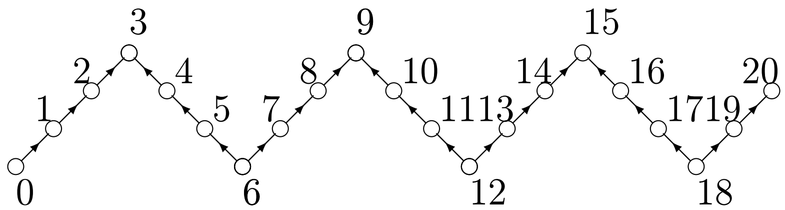

Figure 27

In this case we get the lattice:

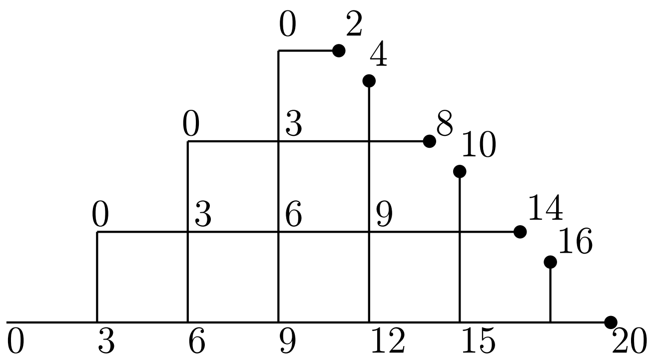

Figure 28

From Figures 10 and 17, we conclude

$$\varepsilon(\vec{P}_{21;3}, 0) = \varepsilon(P_8, 0) = 2\,\varepsilon(P_7, 0) - M_0(3, 3)\ .$$

For $\varepsilon(\vec{P}_{21;3}, 6)$, we have:

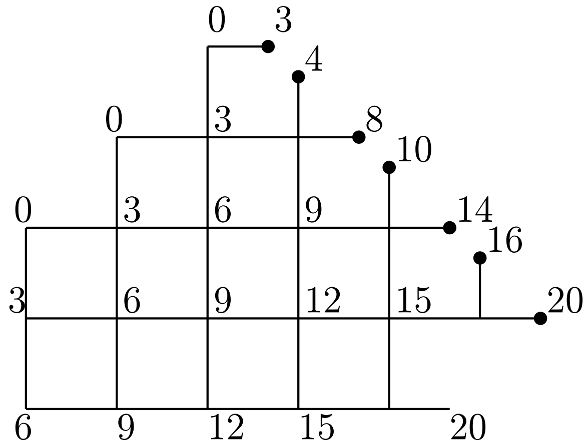

Figure 29

and, from Figure 13, with $n = 4$,

$$\varepsilon(\vec{P}_{21;3}, 6) = 2\,\varepsilon(P_7, 2) - M_2(2, 4)\ .$$

Analogously,

$$\varepsilon(\vec{P}_{21;3}, 12) = 2\,\varepsilon(P_7, 4) - M_4(1, 5)$$

and

$$\varepsilon(\vec{P}_{21;3}, 18) = 2\,\varepsilon(P_7, 6) - M_6(0, 6)\ .$$

Thus, motivated by these examples we can easily establish the following equality when n is even in (6.1):

$$\varepsilon(\vec{P}_{m;k}, t) = 2\,\varepsilon\left(P_{n+1}, \frac{t}{k}\right) - M_{\frac{t}{k}}\left(\frac{n}{2} - \frac{t}{2k}, \frac{n}{2} + \frac{t}{2k}\right)\ .$$

Acknowledgment

We are grateful to Professor Ulrich Knauer, from the Institut für Mathematik, Carl von Ossietzky Universität, Oldenburg, Germany, for his insightful suggestions and comments which have improved the original version.

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