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THE ULTRAFILTER CLOSURE IN ZF

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ABSTRACT: It is well known that, in a topological space, the open sets can be characterized using filter convergence. In ZF (*Zermelo-Fraenkel set theory without the Axiom of Choice*), we cannot replace filters by ultrafilters. It is proven that the ultrafilter convergence determines the open sets for every topological space if and only if the *Ultrafilter Theorem* holds. More, we can also prove that the Ultrafilter Theorem is equivalent to the fact that $u_X = k_X$ for every topological space X, where k is the usual Kuratowski Closure operator and u is the Ultrafilter Closure with

$$u_X(A) := \{x \in X : (\exists \mathcal{U} \text{ ultrafilter in } X)[\mathcal{U} \text{ converges to } x \text{ and } A \in \mathcal{U}]\}.$$

However, it is possible to built a topological space X for which $u_X \neq k_X$, but the open sets are characterized by the ultrafilter convergence. To do so, it is proved that if every set has a free ultrafilter then the Axiom of Countable Choice holds for families of non-empty finite sets.

It is also investigated under which set theoretic conditions the equality $u = k$ is true in some subclasses of topological spaces, such as metric spaces, second countable T_0-spaces or $\{\mathbb{R}\}$.

KEYWORDS: Ultrafilter Theorem, Ultrafilter Closure.
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1. Introduction

In a topological space, the closure is characterized by the limits of the ultrafilters. Although, in the absence of the Axiom of Choice, this is not a fact anymore. The first goal of this paper is to find out the set-theoretic status of this theorem of ZFC, i.e., Zermelo-Fraenkel set theory including the Axiom of Choice. It is also natural to ask if it is equivalent to define the closure or the open sets via ultrafilter convergence. In other words, if the following two theorems are equivalent in ZF, Zermelo-Fraenkel set theory without the Axiom of Choice.

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Theorems 1.1 (ZFC)**.**

(a) *The point $x \in X$ is in the closure of $A \subseteq X$ if and only if there is an ultrafilter $\mathcal{U}$ in X such that $\mathcal{U}$ converges to x and $A \in \mathcal{U}$.*

(b) *The set $A \subseteq X$ is open if and only if for every ultrafilter $\mathcal{U}$ in X converging to $x \in A$, A belongs to $\mathcal{U}$.*

To better understand and study the problem, one consider the *Ultrafilter Closure* operator u defined by limits of ultrafilters and its idempotent hull $\hat{u}$.

Definitions 1.2. Let A be a subspace of the topological space X.

(a) $u_X(A) := \{x \in X : (\exists \mathcal{U} \text{ ultrafilter in } X)[\mathcal{U} \text{ converges to } x \text{ and } A \in \mathcal{U}]\}$.

(b) $\hat{u}_X(A) := \bigcap\{B : A \subseteq B \text{ and } u_X(B) = B\}$.

Denoting by k the usual Kuratowski closure operator, it is easy to see that in ZF for every $A \subseteq X$, $u_X(A) \subseteq \hat{u}_X(A) \subseteq k_X(A)$.

It is now possible to rewrite Theorems 1.1. The first theorem just says that $u_X = k_X$ for every topological space X. And the second says that $\hat{u}_X = k_X$ for every topological space X. As we will see in Theorem 3.1, both conditions are equivalent to the *Ultrafilter Theorem.* So one can ask

1. Is there a topological space X for which $\hat{u}_X = k_X$ but $u_X \neq k_X$?

We will also investigate in which conditions $u = \hat{u}$, or equivalently ask

2. Is the Ultrafilter Closure idempotent?

It is not surprising that the answers to questions 1 and 2 are linked. That is, they are the opposite as the next proposition shows.

Proposition 1.3 (ZF)**.** *The Ultrafilter Closure is idempotent for every topological space if and only if for every topological space $(X, \mathcal{T})$, $\hat{u}_X = k_X$ implies $u_X = k_X$.*

Proof: If u_X is idempotent, i.e. $u_X = \hat{u}_X$, and $\hat{u}_X = k_X$ then $u_X = k_X$.

Let $(X, \mathcal{T})$ be a topological. Define now a topological space $(Y, \mathcal{S})$ with $Y = X$ and $\mathcal{S} := \{Y \setminus A : u_X(A) = A\}$. With these topologies, the equalities $u_X = u_Y$ and $\hat{u}_Y = k_Y$ hold. Since $\hat{u}_Y = k_Y$ implies $u_Y = k_Y$, and k_Y is idempotent, u_X is also idempotent. ■

Note that in general $\hat{u}_X \neq k_X$, and then there was the need to use the space Y.

We will also study when each of the equalities $u = k$, $\hat{u} = k$ and $u = \hat{u}$ hold in the classes of: topological spaces, Hausdorff spaces, first countable spaces, metric spaces, spaces with countable topologies, second countable T_0-spaces, subspaces of $\mathbb{R}$ and $\{\mathbb{R}\}$.

Let us notice that, in ZF, the situation with the spaces defined by the ultrafilter convergence can be compared with the spaces defined by the sequential convergence ([3]).

In order to work in the class of second countable T_0-spaces, it is important to recall the following result.

Lemma 1.4.

(a) *If $(X, \mathcal{T})$ is a second countable space, then $|\mathcal{T}| \leq |\mathbb{R}| = 2^{\aleph_0}$.*
(b) *If $(X, \mathcal{T})$ is a second countable T_0–space, then $|X| \leq |\mathbb{R}| = 2^{\aleph_0}$.*

2. Consequences of the Axiom of Choice

In this section, we introduce some definitions and results related with consequences of the Axiom of Choice which will be used in the other sections. From this section, all definitions and results of this paper take place in ZFA. ZFA is a weaker version of ZF which allows the presence of *atoms*.

The ZFC results of the first section can be proved using the Ultrafilter Theorem which is known to be properly weaker then the Axiom of Choice (Basic Cohen Model – $\mathcal{M}1$ in [6]).

Definition 2.1 (Ultrafilter Theorem – **UFT**). Every filter over a set can be extended to an ultrafilter.

This theorem is equivalent to the *Boolean Prime Ideal Theorem*, that is: every non-trivial ($0 \neq 1$) Boolean Algebra has a prime ideal. For details on this subject, see [7, 2.3] or [4, 4.37].

We consider some weak forms of the Ultrafilter Theorem.

Definitions 2.2.

(a) ([5]) (**Countable Ultrafilter Theorem** – **CUF**) The Ultrafilter Theorem holds for filters with a countable base.
(b) **CUF($\mathbb{R}$)** states that the Ultrafilter Theorem holds for filters in $\mathbb{R}$ with a countable base.

Proposition 2.3. *If the Axiom of Countable Choice (***CC***) holds and $\mathbb{N}$ has a free ultrafilter, then* **CUF** *holds.*

Proof: Let $(A_n)_n$ be a base for a filter in a set X. One can consider that $A_{n+1} \subsetneq A_n$ for every n. By **CC** we choose an element a_n of $A_n \setminus A_{n+1}$. By hypothesis, $\mathbb{N}$ has a free ultrafilter and then the set $\{a_n : n \in \mathbb{N}\}$ has a free ultrafilter $\mathcal{U}$. The filter generated in X by $\mathcal{U}$ is an ultrafilter which contains $\{A_n : n \in \mathbb{N}\}$. ∎

Corollary 2.4. *The Ultrafilter Theorem is not equivalent to* **CUF**.

In Pincus Model IX ($\mathcal{M}47(n, M)$ in [6]) the Ultrafilter Theorem does not hold, but the Axiom of Countable Choice holds and there is a free ultrafilter in $\mathbb{N}$, and then **CUF**$(\mathbb{R})$ holds too.

Proposition 2.5. *If the Axiom of Choice holds for families of subsets of* $\mathbb{R}$ *(***AC**$(\mathbb{R})$*), then* **CUF**$(\mathbb{R})$ *holds.*

As in the proof of Proposition 2.3, if the Axiom of Countable Choice holds for subsets of $\mathbb{R}$ (**CC**$(\mathbb{R})$) and $\mathbb{N}$ has a free ultrafilter, then **CUF**$(\mathbb{R})$ holds. Clearly **AC**$(\mathbb{R})$ implies **CC**$(\mathbb{R})$ and, since $|\mathbb{R}| = |2^{\mathbb{N}}|$, **AC**$(\mathbb{R})$ also implies that every filter in $\mathbb{N}$ can be extended to an ultrafilter.

Corollary 2.6. **CUF** *is not equivalent to* **CUF**$(\mathbb{R})$.

In The Second Fraenkel Model ($\mathcal{N}2$ in [6]) **CUF** does not hold, but the Axiom of Choice holds in $\mathbb{R}$, and then **CUF** holds too.

It is maybe surprising, but of importance in this context, that the existence of a free ultrafilter in $\mathbb{R}$ implies the existence of a free ultrafilter in $\mathbb{N}$.

Theorem 2.7 ([2])**.** $\mathbb{R}$ *has a free ultrafilter if and only if* $\mathbb{N}$ *has a free ultrafilter.*

Proof: If $\mathbb{N}$ has a free ultrafilter, then that ultrafilter can be extended to an ultrafilter in $\mathbb{R}$.

Consider now $\mathcal{U}$ an ultrafilter in $\mathbb{R}$. It is a theorem of ZFA that every ultrafilter in $\mathbb{R}$ is either convergent or unbounded. Without loss of generality, one supposes that $\mathcal{U}$ has no upper bound, that is, for every $x \in \mathbb{R}$, $(x, +\infty) \in \mathcal{U}$. Let $(B_n)_n$ be the partition of $\mathbb{R}$ with $B_1 := (-\infty, 1]$ and $B_n := (n-1, n]$ for $n \in \mathbb{N} \setminus \{1\}$.

The set $\mathcal{U}' := \{A \subseteq \mathbb{N} : \bigcup_{n\in A} B_n \in \mathcal{U}\}$ is a free ultrafilter in $\mathbb{N}$. ∎

This result allows us to better understand the existence of free ultrafilters in $\mathbb{R}$, since there are several known models of ZF where $\mathbb{N}$ has no free ultrafilters.

For instance the Feferman's Model – $\mathcal{M}2$ in [6]. For more details, see Form 70 of [6].

The last result of this section will be useful to find a topological space where the Ultrafilter Closure is not idempotent.

Theorem 2.8. *If every set has a free ultrafilter, then the Axiom of Countable Choice holds for families of finite sets.*

Proof: Let $(X_n)_{n\in\mathbb{N}}$ be a countable family of non-empty finite sets. For each n, define $Y_n := \prod_{k=1}^{n} X_n$ and let $Y := \dot{\bigcup} Y_n$ be their disjoint union. For every $n \in \mathbb{N}$ and $k \leq n$, consider the kth projection $p_k : Y_n \to X_k$. Since p_k is defined in Y_n for every $n \geq k$, p_k is a function from $Z_k := \bigcup_{n=k}^{\infty} Y_n$ to X_k.

There is an equivalence relation in Z_k, for each k, with $y \sim_k y' :\Longleftrightarrow p_k(y) = p_k(y')$. The number of equivalence classes of the relation $\sim_k$ is equal to the number of elements of X_k, which is finite.

Our assumption says that there is a free ultrafilter $\mathcal{U}$ in Y. Since all the sets Y_n are finite, the sets $Y \setminus Z_k$ are finite and then $Z_k \in \mathcal{U}$. For each $k \in \mathbb{N}$, Z_k is the finite union of the equivalence classes of the relation $\sim_k$. By the definition of ultrafilter one, and only one, of the classes belongs to $\mathcal{U}$. Let us call that class A_k.

Finally, we define $x_k := p_k(y)$ for $y \in A_k$. The family $(x_n)_{n\in\mathbb{N}}$ induces the desired choice function. ■

Remark. Recall that the Ultrafilter Theorem implies the Axiom of Choice for families of finite sets, see for instance [4, 2.16].

3. Main results

In this section we investigate under which conditions the Theorems 1.1 and the idempotency of the Ultrafilter Closure remain valid.

Theorem 3.1. *The closure operators $\hat{u}$ and k coincide in the class of Hausdorff spaces if and only if the Ultrafilter Theorem(***UFT***) holds.*

Proof: If **UFT** holds, then the usual proof works.

To prove the reverse implication, let $\mathcal{F}$ be a free filter in a set X and a an element of X. One takes X with the topology generated by the basis

$\mathcal{B} := \{\{x\} : x \in X \setminus \{a\}\} \cup \{F \cup \{a\} : F \in \mathcal{F}\}$. This topology is an Hausdorff topology.

The filter $\mathcal{F}$ is free which implies that there is an element B in $\mathcal{F}$ such that $a \notin B$. It is clear that $k_X(B) = B \cup \{a\}$. Since $k_X(B) = \hat{u}_X(B) = B \cup \{a\}$, there is an ultrafilter $\mathcal{U}$ converging to a such that $B \in \mathcal{U}$. To complete the proof, one only has to show that $\mathcal{F}$ is contained in $\mathcal{U}$.

For every set F in $\mathcal{F}$, $F \cup \{a\} \in \mathcal{U}$ since $\mathcal{U}$ converges to a. One also has that $B \in \mathcal{U}$, and then $B \cap (F \cup \{a\}) = B \cap F \in \mathcal{U}$, which means that F is an element of $\mathcal{U}$. ■

Corollary 3.2. *The following conditions are equivalente to* **UFT***:*

(i) $u = k$ *in the class of topological spaces;*
(ii) $\hat{u} = k$ *in the class of topological spaces;*
(iii) $u = k$ *in the class of Hausdorff spaces.*

The conditions (i) and (ii) of this corollary are the Theorems 1.1.

We will now study the idempotency of the Ultrafilter Closure u, or equivalently the equality $u = \hat{u}$. However, in this case there is no definitive answer.

At this point one could believe that u and $\hat{u}$ coincide in the class of topological spaces if and only if the Ultrafilter Theorem holds. That is not the case as we will see.

Lemma 3.3. *For every topological space $(X, \mathcal{T})$ such that X has no free ultrafilters and for all $A \subseteq X$:*

$$u_X(A) = \bigcup_{a \in A} k_X(\{a\}).$$

If $(X, \mathcal{T})$ is a T_1-space, then $u_X(A) = A$.

There are in fact models of ZF with sets which have no free ultrafilters or, even more, with no free ultrafilters at all. A. Blass [1] built a model ($\mathcal{M}15$ in [6]) where every ultrafilters are fixed. For details, see Forms 63 and 206 of [6].

Proposition 3.4. *If every ultrafilter in X is fixed, then u_X is idempotent.*

Proof: Follow from Lemma 3.3. ■

Corollary 3.5. *It is consistent with ZF that $\hat{u} = u$ in the class of the topological spaces and* **UFT** *does not hold.*

This conditions is true in the Feferman/Blass Model ($\mathcal{M}15$ em [6]).

After seeing that the idempotency of the Ultrafilter Closure is not equivalent to the Ultrafilter Theorem, our goal is to find out if it is a theorem of ZFA.

Theorem 3.6. *Let X, Y be sets such that X has a free ultrafilter and Y has no free ultrafilters. If there is $f : Y \to X$ onto, then there is a topological space for which the Ultrafilter Closure is not idempotent.*

Proof: The existence of $f : Y \to X$ onto means that $Y = \dot{\bigcup_{x\in X}} Y_x$ is the disjoint union of a family of non-empty sets indexed by X. Suppose that X and Y are disjoint and that $\infty \notin X \cup Y$.

Define $Z := X \cup Y \cup \{\infty\}$ and $\mathcal{T} := \{A \cup \bigcup_{x\in A} Y_x \cup \{\infty\} \, : \, X \setminus A \text{ is finite } \}$. The pair $(Z, \mathcal{T})$ is a topological space. Since the ultrafilters in Y are fixed, $u_Z(Y) = X \cup Y$. On the other side, the set X has a free ultrafilter $\mathcal{U}$. If $A \subseteq X$ and $X \setminus A$ is finite, then $A \in \mathcal{U}$. This fact implies that the ultrafilter generated in Z by $\mathcal{U}$ converges to ∞. It is now clear that $u_Z^2(Y) = Z \neq u_Z(Y)$, and one concludes that u_Z is not idempotent. ■

Remark. In this prove, one can replace $\mathcal{T}$ by the T_0 topology

$$\mathcal{T}' := \mathcal{T} \cup \{\{x\} \cup Y_x \, : \, x \in X\} \cup \{\{y\} \, : \, y \in Y\}.$$

The proof works in the same way. From Lemma 3.3 is clear that the same construction cannot be done if the topology is T_1.

Corollary 3.7. *If for every topological space, the Ultrafilter Closure is idempotent, then either the Axiom of Countable Choice holds for families of finite sets (**CC**(fin)) or there are no free ultrafilters in $\mathbb{N}$.*

Proof: Let $(X_n)_n$ be a countable family of non-empty finite sets. Consider the set Y as in the proof of Theorem 2.8. By the way Y was builded, there is a surjective function from Y to $\mathbb{N}$. Since the Ultrafilter Closure is idempotent, the Theorem 3.6 implies that either there are no free ultrafilters in $\mathbb{N}$ or there is a free ultrafilter in Y. As in the proof of Theorem 2.8, the fact that Y has a free ultrafilter implies that there is a choice in $(X_n)_n$, that is **CC**(fin) holds. ■

Corollary 3.8. *It is consistent with* ZFA *that there is a first countable T_0-space Z such that $u_Z \neq u_Z^2$.*

In The Second Fraenkel Model ($\mathcal{N}_2$ in [6]) **CC**(fin) fails and $\mathbb{N}$ has a free ultrafilter. Then the space Z of the proof of Theorem 3.6, with $X = \mathbb{N}$ and $\mathcal{T}'$ as in the remark after the theorem, is a first countable space T_0-space and $u_Z \neq u_Z^2$. If we drop the condition of being T_0, it is clear from the proof of Theorem 3.6 that the topology might be countable.

We arrive to the conclusion that the answers to the two questions of the Introduction are affirmative and negative, respectively.

4. The real numbers

We look now to the restriction of the conditions we have been studied to certain classes of topological spaces. Among them, it is curious to see what is the situation in the topological space $\mathbb{R}$ with the Euclidian Topology.

Proposition 4.1. *The following conditions are equivalente to* **CUF***:*

(i) *$u = k$ in the class of first countable spaces;*
(ii) *$\hat{u} = k$ in the class of first countable spaces;*
(iii) *$u = k$ in the class of metric spaces;*
(iv) *$\hat{u} = k$ in the class of metric spaces.*

Proof: The implications (i)⇒(ii)⇒(iv) and (i)⇒(iii)⇒(iv) are obvious. It is enough to show that **CUF**⇒(i) and (iv)⇒**CUF**.

CUF⇒(i) Let X be a first countable space, $A \subseteq X$ and $x \in k_X(A)$. The space X is first countable, which means that x has countable neighborhood base $(V_n)_n$. The set $\{V_n \cap A : n \in \mathbb{N}\}$ is a base for a filter in A. This filter converges to x, since $x \in k_X(A)$.

Finally, by hypothesis, every filter with countable base can be extended to an ultrafilter. This ultrafilter contains A and converges to x, that is $x \in u_X(A)$.

(iv)⇒**CUF** Let $(A_n)_n$ be a base for a free filter in X. Without lost of generality, one consider $A_{n+1} \subseteq A_n$ for every n and $A_1 \neq X$. Let $a \in X \setminus A_1$ and redefine $A_1 := X \setminus \{a\}$. The family $(A_n)_n$ is still a base for the same filter.

Define the sets $B_n := A_n \setminus A_{n+1}$, for each $n \in \mathbb{N}$. Since the filter is free, $X \setminus \{a\} = \bigcup\limits_n B_n$.

One has the following metric on X:

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } (x = a \text{ and } y \in B_n) \text{ or } (x \in B_n \text{ and } y = a) \\ \frac{1}{n} + \frac{1}{m} & \text{if } x \in B_n \text{ and } y \in B_m. \end{cases}$$

The element a is in the closure of A_1 and then A_1 is not closed. By (iv), there is an ultrafilter $\mathcal{U}$ in X such that $A_1 \in \mathcal{U}$ and $\mathcal{U}$ converges to an element not belonging to A_1. Since $X \setminus A_1 = \{a\}$, $\mathcal{U}$ converges to a.

For every n, the sets $A_n \cup \{a\}$ are neighborhoods of a, which implies that they are in $\mathcal{U}$. Consequently, for every n, $A_n = (A_n \cup \{a\}) \cap A_1 \in \mathcal{U}$, which means that $\mathcal{U}$ extends the filter generated by (A_n). ■

Proposition 4.2. *The following conditions are equivalent to* **CUF***:*

(i) $u = k$ *in the class of spaces with countable topologies;*
(ii) $\hat{u} = k$ *in the class of spaces with countable topologies.*

Proof: After Proposition 4.1, only remains to be proved that (ii)⇒**CUF** .

Let $(A_n)_n$, a and X be as in the proof of Proposition 4.1. We give a countable topology on X,

$$\mathcal{T} := \{A_n \cup \{a\} : n \in \mathbb{N}\}.$$

From this point, the proof follows as in the proof of 4.1. ■

Theorem 4.3. *The closure operators u and k coincide in $\mathbb{R}$ if and only if* **CUF($\mathbb{R}$)** *does hold.*

Proof: In Proposition 4.1, it was shown that if **CUF** holds, then for every first countable space X, $u_X = k_X$. From that proof, it is clear that it is only necessary to apply **CUF** in X. One can conclude that **CUF($\mathbb{R}$)** suffices to prove that $u_{\mathbb{R}} = k_{\mathbb{R}}$.

Let $\mathcal{F}$ be a free ultrafilter in $\mathbb{R}$ with a countable base. One can consider a base $(A_n)_n$ of $\mathcal{F}$ such that $A_{n+1} \subseteq A_n$ for every n. Define $B_n := A_n \setminus A_{n+1}$. There are bijective functions, in ZFA, $f_n : \mathbb{R} \longrightarrow (\frac{1}{n+1}, \frac{1}{n})$ for every $n \in \mathbb{N}$. Define also $C_n := f_n(B_n) \subseteq (\frac{1}{n+1}, \frac{1}{n})$ and $D := \bigcup_n C_n$. The filter $\mathcal{F}$ is free, which implies that $0 \in k_{\mathbb{R}}(D)$. Since $u_{\mathbb{R}} = k_{\mathbb{R}}$, $0 \in u_{\mathbb{R}}(D)$ and then there is a free ultrafilter $\mathcal{U}$ converging to 0 such that $D \in \mathcal{U}$.

Let $\mathcal{U}'$ be the restriction of $\mathcal{U}$ to D. The function $t : D \longrightarrow A_1$ with $t(x) = f_n^{-1}(y)$ if $x \in C_n$ is well-defined and it is bijective due to the way the functions f_n and the sets C_n and D were defined. One can conclude that $t(\mathcal{U}')$ is an ultrafilter in A_1.

Since $\mathcal{U}$ converges to 0, $(-\frac{1}{n}, \frac{1}{n}) \in \mathcal{U}$ for every $n \in \mathbb{N}$ and then $\bigcup_{k=n}^{\infty} C_k = D \cap (-\frac{1}{n}, \frac{1}{n}) \in \mathcal{U}'$.

Finally, we have that

$$A_n = \bigcup_{k=n}^{\infty} B_k = \bigcup_{k=n}^{\infty} f_n^{-1}(C_k) = \bigcup_{k=n}^{\infty} t(C_k) = t(\bigcup_{k=n}^{\infty} C_k) \in t(\mathcal{U}')$$

and the ultrafilter generated in $\mathbb{R}$ by $t(\mathcal{U}')$ contains $\mathcal{F}$. ∎

Corollary 4.4. *The following conditions are equivalente to* $\mathbf{CUF(\mathbb{R})}$*:*

(i) $u = k$ *in the class of the second countable* T_0*-spaces;*
(ii) $\hat{u} = k$ *in the class of the second countable* T_0*-spaces;*
(iii) $u = k$ *for subspaces of* $\mathbb{R}$*;*
(iv) $\hat{u} = k$ *for subspaces of* $\mathbb{R}$*.*

Proof: It is clear that (i)⇒(ii)⇒(iv) and (i)⇒(iii)⇒(iv).

$\mathbf{CUF(\mathbb{R})}$⇒(i) By Lemma 1.4, if X is a T_0 space with a countable base, then $|X| \leq |\mathbb{R}|$. Since it is enough to apply **CUF** in X, $\mathbf{CUF(\mathbb{R})}$ suffices to show that $u_X = k_X$.

(iv)⇒$\mathbf{CUF(\mathbb{R})}$ The proof is identical to the proof of Theorem 4.3. Let D be a set constructed as in proof of Theorem 4.3 and $X := D \cup \{0\}$ be a subspace of the reals. One has that $0 \in k_X(D)$, and then, by (iv), one also has that $0 \in \hat{u}_X(D)$. Since $X \setminus D = \{0\}$, $0 \in u_X(D)$ e consequently $0 \in u_{\mathbb{R}}(D)$. From now on, the proof follows as in the proof of Theorem 4.3. ∎

Proposition 4.5. *If* $\mathbb{R}$ *has no free ultrafilters, then* $\hat{u}_{\mathbb{R}} \neq k_{\mathbb{R}}$*.*

This proposition is straightforward after Lemma 3.3. $\mathbb{R}$ is a T_1-space, and then $u_{\mathbb{R}}$ is discrete if $\mathbb{R}$ has no free ultrafilters.

From this proposition, one concludes that $\hat{u}_{\mathbb{R}} = k_{\mathbb{R}}$ is not a theorem of ZF.

There are models of ZF where $\mathbb{R}$ has no free ultrafilters, but other sets have. The Feferman's Model is an example of that.

I do not know if the equality $\hat{u}_{\mathbb{R}} = k_{\mathbb{R}}$ is properly weaker than $\mathbf{CUF(\mathbb{R})}$.

As in the general case, the situation of the idempotency of u is different from the other equalities. The following results are just additions to the corollaries 3.5 and 3.8.

Proposition 4.6. *The following sentences are consistent with* $\mathsf{ZF0}$.

(a) $\hat{u} = u$ *in the class of first countable spaces and* **CUF** *does not hold.*

(b) $\hat{u}_{\mathbb{R}} = u_{\mathbb{R}}$ *and* **CUF**$(\mathbb{R})$ *does not hold.*

(c) $\hat{u} \neq u$ *in the class of spaces with countable topologies.*

Similar conclusions could be written for other classes.

Another interesting question is to find out if $u_{\mathbb{R}}$ is idempotent in ZFA. Clearly, if $u_{\mathbb{R}}$ is idempotent, then $\hat{u}_{\mathbb{R}} = k_{\mathbb{R}}$ is equivalent to **CUF**$(\mathbb{R})$, by Theorem 4.3.

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