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CUBIC POLYNOMIALS AND OPTIMAL CONTROL ON COMPACT LIE GROUPS

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Abstract: This paper analyzes the Riemannian cubic polynomials's problem from a Hamiltonian point of view. The description of the problem on compact Lie groups is particulary explored. The state space of the second order optimal control problem considered is the tangent bundle of the Lie group which also has a group structure. The dynamics of the problem is described by a presymplectic formalism associated with the canonical symplectic form on the cotangent bundle of the tangent bundle. Using these control geometrical tools, the equivalence between the Hamiltonian approach developed here and the known variational one is verified. Moreover, the equivalence allows us to deduce two invariants along the cubic polynomials which are in involution.

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1. Introduction

Riemannian cubic polynomials (RCP) can be seen as a generalization of cubic polynomials in Euclidean spaces to Riemannian manifolds. The cubic polynomials on a Riemannian manifold are the smooth solutions of a fourth order differential equation which is the Euler-Lagrange equation of a second order variational problem. This variational problem was first introduced in 1989 (see [17]) and explored from a dynamical interpolation perspective in 1995 (see [10]). Interesting points related to this subject have been developed in the last few years, namely a geometric theory surprisingly close to the Riemannian theory of geodesics (see [2, 3, 4, 7, 8, 9, 16, 18, 19]). More recently, in [3, 16, 18], the analysis of RCP from a variational point of view was carried out for locally symmetric manifolds and some invariants along these cubic polynomials were obtained. A qualitative analysis of RCP is given in [3, 16], with special attention to the case of the Lie group $SO(3)$, where RCP corresponds to Lie quadratics on the Lie algebra. The article [3] introduces a reduction of the RCP equation for this Lie group of rotations. In [16] some results on asymptotics and symmetries of cubics are proved for

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the particular case of the so-called null cubic polynomials on $SO(3)$. Finally, [18] studies n-th generalizations of RCP.

A Hamiltonian perspective of the RCP problem was first introduced in [8]. The present paper gives a different description of the problem on arbitrary compact Lie groups. We use here a presymplectic approach to the Pontryagin's Maximum Principle based on some control geometric ideas of [4, 5, 6, 14, 11]. The new contribution of this work is to use the Lie group structure of the tangent bundle of a Lie group, which in this case is the state space of the optimal control problem. This allows us to use classical results from [1, 12], adapted to this tangent Lie group structure.

The paper is organized as follows.

Section 2 presents the RCP's variational problem and the most relevant results, namely the deduction of the two known invariants for locally symmetric manifolds. We finish the section recalling the equivalent RCP's optimal control problem introduced in [4].

The presymplectic approach of this optimal control problem in an arbitrary Riemannian manifold is the subject of section 3. This viewpoint was inspired by some ideas of [14] and [11], where an intrinsic geometric approach for a first order general optimal control problem is considered, by means of a presymplectic description of the dynamical system. In [5] a similar geometric approach is also considered for time-dependent optimal control problems by using the jet bundles framework.

The last section contains the main results of the paper: the analysis of RCP on compact Lie groups. The system under consideration will be thus defined on the tangent bundle of a Lie group. Such a space can also be endowed with a Lie group structure. Hence, the dynamics on this state space will be described by the presymplectic formalism associated with the canonical symplectic form adapted to the tangent Lie group structure. We thus provide a Hamiltonian description equivalent to the variational one ([10]) in a similar way to what happens in [8]. However, it is important to remark that our Hamiltonian system and the one in [8] are different. In this context, besides the identification of the first invariant as the Hamiltonian of the presymplectic system, we make a geometric deduction of the two invariants.

Throughout the paper we consider an n-dimensional Riemannian manifold M, with Riemannian metric $\langle ., . \rangle$. The symmetric connection on M, which is compatible with this metric, is denoted by ∇ and the covariant derivative along a curve x in M by DY/dt, where Y is a vector field along x. Moreover, we denote the curvature tensor field by R and the covariant differential of R by ∇R.

TM represents the tangent bundle and T^2M the second tangent bundle. If we consider the local coordinates (x^i) on M, the standard local coordinates on TM and T^2M are denoted by (x^i, y^j) and (x^i, y^j, u^l), respectively. Consider also (x^i, y^j, v^k, u^l) and (x^i, y^j, p_k, q_l) to be the corresponding coordinates on TTM and T^*TM, respectively.

In the last section, $M = G$ represents a compact Lie group.

2. Cubic polynomials on Riemannian manifolds

The aim of this section is to introduce the Riemannian cubic polynomials. We present the variational problem which gave rise to the first studies of the RCP and deduce, for locally symmetric manifolds, two invariant along these particular curves. We finish the section formulating the equivalent RCP's optimal control problem.

2.1. Riemannian cubic polynomials's variational problem. In order to generalize the notion of cubic polynomials to a Riemannian manifold, the following second order variational problem in M was formulated in [10, 17]:

$$\min_{x \in \mathcal{C}} \frac{1}{2} \int_0^T \left\langle \frac{D^2x}{dt^2} \, , \, \frac{D^2x}{dt^2} \right\rangle dt,$$

where $\mathcal{C}$ is the class of C^1 piecewise smooth curves $x : [0, T] \to M$, satisfying

$$x(0) = x_0, \quad \frac{dx}{dt}(0) = y_0, \quad x(T) = x_T, \quad \frac{dx}{dt}(T) = y_T,$$

with $y_0 \in T_{x_0}M$, $y_T \in T_{x_T}M$, $x_0, x_T \in M$ and $T \in \mathcal{R}^+$.

Definition 1. *We call Riemannian cubic polynomials on M to the smooth curves $x : [0, T] \to M$ solutions of the Euler-Lagrange equation of the above problem, that is, the fourth order differential equation*

$$\frac{D^4x}{dt^4} + R\left(\frac{D^2x}{dt^2} \, , \, \frac{dx}{dt}\right)\frac{dx}{dt} = 0. \tag{1}$$

Let x be such a solution and let us denote its velocity vector field, $\frac{dx}{dt}$, by V.

Proposition 1. [3] *In locally symmetric manifolds, the two following expressions are invariants along the cubic polynomial:*

$$I_1 = \frac{1}{2}\left\langle \frac{DV}{dt}, \frac{DV}{dt} \right\rangle - \left\langle \frac{D^2V}{dt^2}, V \right\rangle$$

and

$$I_2 = \left\langle \frac{D^2V}{dt^2}, \frac{D^2V}{dt^2} \right\rangle - \left\langle \frac{D^3V}{dt^3}, \frac{DV}{dt} \right\rangle.$$

Proof: The invariance of the first expression follows from the integration of the inner product of (1) with V. To prove the second one is also invariant, we make the inner product of (1) with $\frac{D^2V}{dt^2}$. In the resulting equation, apply the tensor curvature property $\langle R(X,Y)Z,W\rangle = \langle R(W,Z)Y,X\rangle$ and the definition of the covariant differentiation of the curvature tensor to get

$$\frac{d}{dt}\left[\left\langle \frac{D^2V}{dt^2}, \frac{D^2V}{dt^2} \right\rangle - \left\langle \frac{D^3V}{dt^3}, \frac{DV}{dt} \right\rangle\right] = \left\langle (\nabla_V R)\left(\frac{DV}{dt}, V\right)V, \frac{DV}{dt} \right\rangle.$$

Now the result follows from the fact that $\nabla R \equiv 0$ in locally symmetric manifolds. ■

Note that I_1 is invariant in any Riemannian manifold.

2.2. Riemannian cubic polynomials's optimal control problem. The optimal control problem corresponding to the RCP's variational problem is now considered. The idea was explained in [4], where we associated a second order control system to the RCP.

In general, to formulate a second order optimal control problem, we can consider the control system defined by a vector field $\Pi : T^2M \to TTM$ along the natural projection map $\tau_2^1 : T^2M \to TM$. Hence, TM shall be the state space and T^2M the control bundle.

$$\begin{tikzcd} T^2M \arrow[r, "\Pi"] \arrow[dr, "\tau_2^1"'] & TTM \arrow[d, "\pi_{TM}"] \\ & TM \end{tikzcd}$$

Optimal control problem: find the C^2 piecewise smooth curve $\gamma : [0,T] \to T^2M$ with fixed endpoints in the state space, satisfying the control system

$$\frac{d}{dt}\left(\tau_2^1(\gamma(t))\right) = \Pi(\gamma(t)) \qquad (2)$$

and minimizing the functional $\int_0^T L(\gamma(t))dt$, where $L : T^2M \to \mathcal{R}$ is the cost function, with fixed $T \in \mathcal{R}^+$.

We wil define a suitable control system to our case. Let $T\pi_{_M} : TTM \to TM$ be the differential of the natural projection $\pi_{_M}$ of TM onto M locally defined by $T\pi_{_M}(x^i, y^i, v^i, u^i) = (x^i, v^i)$ and consider the so called connection map defined in what follows.

Definition 2. *For each point $(v, u) \in T_{(x,y)}TM$ with $(x, y) \in TM$, let α be the curve in TM with initial value (x, y) and initial velocity (v, u). The connection map is the smooth map $K : TTM \to TM$ defined by*

$$K(x, y, v, u) = K_{|_{(x,y)}}(v, u) = \nabla_{\frac{d(\pi_M \circ \alpha)(t)}{dt}} \alpha(t)_{|_{t=0}}.$$

In local coordinates, $K(x^i, y^i, v^i, u^i) = (x^i, \{u^i + \sum_{jk=1}^{n} \Gamma^i_{jk} y^j v^k\})$ where Γ^i_{jk} represent the Christoffel symbols of the connection ∇. Consider also an application $J : TM \to TM$, locally defined by $J(x^i, y^i) = (x^i, \{\sum_{j,k=1}^{n} \Gamma^i_{jk} y^j y^k\})$.

Now we have the tools to describe the control system associated to the RCP problem. With the above definitions, the control system corresponds to (2) with a mapping Π which satisfies

$$T\pi \circ (d_T - \Pi) = 0 \qquad \text{and} \qquad K \circ (d_T - \Pi) = J \circ \tau^1_2,$$

where $d_T : T^2M \to TTM$ represents the total time derivative which assigns to each $(x, y, u) \in T^2M$ the element $(x, y, y, u) \in TTM$. The vector field Π is affine in the controls.

To write the cost functional we choose the function on T^2M as

$$L(x, y, u) = \frac{1}{2} \left\langle K(\Pi(x, y, u), K(\Pi(x, y, u) \right\rangle,$$

for $(x, y, u) \in T^2M$, which locally gives $L(x^i, y^i, u^i) = \frac{1}{2} \sum_{i,j=1}^{n} g_{ij}(x)\, u^i u^j$, where g_{ij} denote the components of the Riemannian metric.

3. Presymplectic structure - dynamics of the control problem

In this section we introduce a Hamiltonian description of our system with T^*TM as the co-state space. As we have done in [4], we construct a presymplectic structure using geometric tools from control theory for higher order control systems along the lines of [11]. The dynamical system is presented and a presymplectic constraint algorithm is applied.

3.1. The dynamical system. Let us introduce now the presymplectic structure which describes the dynamics of the control problem. Consider the total space

$$\mathcal{T} = T^*TM \times_{TM} T^2M$$

a vector bundle over TM and let $pr_1 : T^*TM \times_{TM} T^2M \to T^*TM$ and $pr_2 : T^*TM \times_{TM} T^2M \to T^2M$ be the canonical projections. A closed two form is defined by the following pull-back:

$$\Omega = (pr_1)_* \omega_0, \tag{3}$$

where ω_0 is the canonical symplectic two-form on T^*TM. Introduce the Hamiltonian function

$$H = \ll pr_1, \Pi \circ pr_2 \gg - L \circ pr_2, \tag{4}$$

where $\ll .,. \gg$ stands for the canonical duality product of vectors and covectors on TM.

Consider then the presymplectic Hamiltonian system $(\mathcal{T}, \Omega, H)$, whose dynamical vector field $\Gamma : \mathcal{T} \to T\mathcal{T}$ is the solution of the dynamical system

$$i_\Gamma \Omega = dH. \tag{5}$$

The presymplectic system gives necessary conditions for extremal trajectories as a geometric version of those given by the maximum principle.

> A curve γ in T^2M is an extremal trajectory of the optimal control problem if there exists a lifting of γ to the total space $\mathcal{T}$ which is an integral curve of a vector field defined by (5).

We are interested thus in finding out the dynamical vector field Γ solution to the equation (5).

3.2. Geometric algorithm of presymplectic systems. We apply the geometric algorithm of presymplectic systems to $(\mathcal{T}, \Omega, H)$. A first constraint submanifold is determined as

$$W_1 = \{z \in \mathcal{T} : dH(z)(X) = 0, \ \forall\, X \in \operatorname{Ker} \Omega(z)\}.$$

If this manifold is still not symplectic the algorithm continues. However, for the RCP problem, W_1 will be symplectic. Indeed, for the case we are interested in, the constraint submanifold W_1 is locally defined by $\frac{\partial H}{\partial u^i} = 0$, $i = 1, ..., n$. Since the vector field Π which describes the control system is affine in controls, we obtain $\frac{\partial^2 H}{\partial u^j \partial u^i} = -\frac{\partial^2 L}{\partial u^j \partial u^i} = -g_{ij}$. So that, this particular optimal control problem is regular. As a result, W_1 turns out to be a symplectic manifold and the restriction Ω_{W_1} of the presymplectic form Ω to W_1 will be non-degenerate. Consequently, the constraint algorithm stops after the first stage and for each fixed endpoints there will be a unique dynamical vector field Γ_{W_1} solution to the equation (5) defined by

$$i_{\Gamma_{W_1}}\Omega_{W_1} = d\left(H_{|W_1}\right). \tag{6}$$

Remark 1. *An interesting problem is to investigate in this context the integrability of the optimal control problem from the point of view of its symmetries, as it has been done for the first order case in* [11, 14].

4. Riemannian cubic polynomials on compact Lie groups

We specialize the results of the sections 2 and 3 to the case where the manifold is a connected and compact Lie group.

The section begins with some notes about relevant definitions and notations in the context of Lie groups. After that, the RCP's variational problem and the RCP's optimal control problem on connected and compact Lie groups are analyzed. The relation between the two approaches is established.

4.1. Definitions, notations and relevant results on Lie groups. Let us begin presenting the basic properties of Lie groups in order to fix notation.

4.1.1. *Basic notations.* The Lie group is represented by G, its identity is denoted by e and the corresponding Lie algebra, equipped with the Lie bracket product $[.,.]$, by $\mathcal{G}$. We represent by $\mathcal{G}^*$ the dual space of $\mathcal{G}$. Furthermore, the elements of G are denoted by x or g and the maps $G \times G \to G,\ (x, g) \mapsto xg$ and $G \to G,\ x \mapsto x^{-1}$ are the multiplication and inversion operations for the Lie group G, respectively. Given $x, g \in G$, let $L_x : G \to G$ and $R_x : G \to G$ be, respectively, the left and right translations by x. The tangent of L_x at g is denoted by $(TL_x)_g$ and $(TL_x)_g^*$ represents its transpose.

The adjoint representation of the Lie group is denoted by Ad, which gives for each $x \in G$ an algebra automorphism defined by $Ad_x = (T(R_{x^{-1}} \circ L_x))_e$. The tangent of Ad at the identity e, known as the adjoint representation of the Lie algebra, is denoted by ad. We have $ad_Y Z = [Y, Z]$ for each $Y, Z \in \mathcal{G}$. The co-adjoint is the map $ad^* : \mathcal{G} \to Aut(\mathcal{G}^*)$ defined, for each $Y \in \mathcal{G}$ and $\xi \in \mathcal{G}^*$, by $ad^*_Y \xi = -\xi \circ ad_Y$.

4.1.2. *Riemannian metric and connection.* We can guarantee the existence of a bi-invariant metric on G because the Lie group is assumed to be connected and compact. This statement and the following result can be found for instance in [15].

Theorem 1. [15] *If G is a Lie group equipped with a bi-invariant metric, the metric connection ∇ and the curvature tensor R associated with that metric are given by*

$$\nabla_Y Z = \frac{1}{2}[Y, Z], \tag{7}$$

$$R(Y, Z)W = -\frac{1}{4}[[Y, Z], W], \tag{8}$$

where Y, Z and W are left invariant vector fields. Furthermore, (7) implies

$$\langle [Y, Z], W \rangle = \langle Y, [Z, W] \rangle . \tag{9}$$

4.1.3. *Left trivializations of TG and T^*G.* We shall call left trivialization of TG to the isomorphism defined as

$$\begin{array}{rccc} \lambda_{_{TG}}: & G \times \mathcal{G} & \longrightarrow & TG \\ & (x, Y) & \longmapsto & (x, (TL_x)_e(Y)) \end{array}.$$

On the other hand, the following vector bundle isomorphism, which is no more than the inverse of the dual one of λ_{TG}, is called the left trivialization of T^*G:

$$\begin{array}{rccc} \lambda_{_{T^*G}}: & G \times \mathcal{G}^* & \longrightarrow & T^*G \\ & (x, \xi) & \longmapsto & (x, (TL_{x^{-1}})_x^*(\xi)) \end{array}.$$

Consider an orthogonal basis $\{A_1, ..., A_n\}$ of the Lie algebra $\mathcal{G}$ and denote by $\{A_1^*, ..., A_n^*\}$ its dual basis, which is a basis of the dual space $\mathcal{G}^*$. The tangent and cotangent spaces to G at x are defined, respectively, by the following left-invariant frame and co-frame:

$$T_x G = \text{span}\,\{A_i(x)\} \quad \text{such that} \quad A_i(x) = (TL_x)_e(A_i)$$

and

$$T_x^* G = \text{span}\,\{A_i^*(x)\} \quad \text{such that} \quad A_i^*(x) = (TL_{x^{-1}})_x^*(A_i^*).$$

Note that, $\lambda_{_{TG}}(x, A_i) = A_i(x)$ and $\lambda_{_{T^*G}}(x, A_i^*) = A_i^*(x)$ for any $x \in G$.

We now proceed with some more notations useful along the next sections. Let Y be a curve in $\mathcal{G}$ defined by $Y = \sum_{i=1}^n y^i A_i$ and ξ a curve in $\mathcal{G}^*$ written as $\xi = \sum_{i=1}^n p_i A_i^*$, where y^i and p_i are smooth functions of time, $i = 1, \ldots, n$. We have omitted the dependence in time to simplify the notation. The following notations are assumed:

$$\dot{Y} = \sum_{i=1}^n \dot{y}^i A_i \quad \text{and} \quad \dot{\xi} = \sum_{i=1}^n \dot{p}_i A_i^*.$$

Given $\xi \in \mathcal{G}^*$, the tangent vector identified with this co-vector by the Riemannian metric will be denoted by $X_\xi \in \mathcal{G}$. That is, $\xi(Y) =< X_\xi, Y >$, $\forall Y \in \mathcal{G}$. With the above notation, it is simple to verify that

$$\dot{X}_\xi = X_{\dot{\xi}}\,, \quad X_{ad_Y^* \xi} = ad_Y X_\xi \quad \text{and} \quad \frac{d}{dt}\left(ad_Y^* \xi\right) = ad_{\dot{Y}}^* \xi + ad_Y^* \dot{\xi}.$$

4.1.4. *Left trivialization of a Hamiltonian system on T^*G.* Here we recall some classic results on Hamiltonian systems from [1, 12]. Considering the pull-back of the canonical symplectic structure ω on T^*G by the left trivialization $\lambda_{_{T^*G}}$, we obtain a symplectic structure on $G \times \mathcal{G}^*$. We represent it by $\tilde{\omega}$ and call this the left trivialization of the canonical symplectic form ω. By proposition 4.4.1 in page 315 of [1], we know the following lemma.

Lemma 1. *Let $(x, \xi) \in G \times \mathcal{G}^*$ and (W_x, μ_ξ), $(W'_x, \mu'_\xi) \in T_{(x,\xi)}(G \times \mathcal{G}^*) \simeq T_x G \times \mathcal{G}^*$. Then,*

$$\begin{aligned}&\tilde{\omega}(x,\xi)((W_x, \mu_\xi), (W'_x, \mu'_\xi)) = -\mu_\xi\left((TL_{x^{-1}})_x(W'_x)\right) + \\ &+\mu'_\xi\left((TL_{x^{-1}})_x(W_x)\right) + \xi([(TL_{x^{-1}})_x(W_x), (TL_{x^{-1}})_x(W'_x)]).\end{aligned}$$

Given a Hamiltonian system (T^*G, ω, f), where f is an arbitrary Hamiltonian function defined in T^*G, consider the corresponding Hamiltonian system $\left(G \times \mathcal{G}^*, \tilde{\omega}, \tilde{f}\right)$ where $\tilde{f} = f \circ \lambda_{_{T^*G}}$ is the corresponding Hamiltonian function defined in $G \times \mathcal{G}^*$. The Hamiltonian vector field solution of the system is the vector field $\Gamma_{\tilde{f}}$ on $G \times \mathcal{G}^*$ defined by $i_{\Gamma_{\tilde{f}}} \tilde{\omega} = d\tilde{f}$.

Let $(x, \xi) \in G \times \mathcal{G}^*$. In what follows, we denote by $\frac{\partial \tilde{f}}{\partial x}$ the differential of the restriction of $\tilde{f}$ to the set of points in $G \times \mathcal{G}^*$ where ξ is constant, and by $\frac{\partial \tilde{f}}{\partial \xi}$ the differential of the restriction of $\tilde{f}$ to the set of points in $G \times \mathcal{G}^*$ where x is constant. So, $\frac{\partial \tilde{f}}{\partial x}(x, \xi) \in T_x^* G$ and $\frac{\partial \tilde{f}}{\partial \xi}(x, \xi) \in T_\xi^* \mathcal{G}^* \simeq \mathcal{G}$. Given $V : G \times \mathcal{G}^* \to T\left(G \times \mathcal{G}^*\right)$ a vector field on $G \times \mathcal{G}^*$, we can decompose the element $V(x, \xi) = (W_x, \mu_\xi) \in T_{(x,\xi)}\left(G \times \mathcal{G}^*\right) \simeq T_x G \times \mathcal{G}^*$. In such a way,

$d\tilde{f}(x,\xi)$ is an element of $T^*_{(x,\xi)}(G\times\mathcal{G}) \simeq T^*_xG\times T^*_\xi\mathcal{G}^* \simeq T^*_xG\times\mathcal{G}$ and

$$d\tilde{f}(x,\xi)(V(x,\xi)) = \left(\frac{\partial\tilde{f}}{\partial x}(x,\xi)\right)(W_x) + \mu_\xi\left(\frac{\partial\tilde{f}}{\partial\xi}(x,\xi)\right). \tag{10}$$

Theorem 2. *Let $\Gamma_{\tilde{f}}$ be the solution vector field decomposed, for each $(x,\xi)\in G\times\mathcal{G}^*$, as $\Gamma_{\tilde{f}}(x,\xi) = (X,\mu)\in T_xG\times\mathcal{G}^*$. Then,*

$$X = (TL_x)_e\left(\frac{\partial\tilde{f}}{\partial\xi}(x,\xi)\right)$$

and

$$\mu = -(TL_x)^*_e\left(\frac{\partial\tilde{f}}{\partial x}(x,\xi)\right) - ad^*_{(TL_{x^{-1}})_x(X)}\xi.$$

Proof: The solution vector field satisfies the dynamical system $i_{\Gamma_{\tilde{f}}}\tilde{\omega} = d\tilde{f}$, so, for each vector field V on $G\times\mathcal{G}^*$ and each point (x,ξ), we have

$$d\tilde{f}(x,\xi)(V(x,\xi)) = \tilde{\omega}(x,\xi)(\Gamma_{\tilde{f}}(x,\xi), V(x,\xi)).$$

Let $V(x,\xi) = (W_x,\mu_\xi)$, the first member of the above equality can be written as (10). The second one can be expressed as follows using lemma 1,

$$\tilde{\omega}(x,\xi)((X,\mu),(W_x,\mu_\xi)) = -\mu\left((TL_{x^{-1}})_x(W_x)\right) +$$
$$+\mu_\xi\left((TL_{x^{-1}})_x(X)\right) + \xi([(TL_{x^{-1}})_x(X),(TL_{x^{-1}})_x(W_x)]).$$

Then,

$$\left(\tfrac{\partial\tilde{f}}{\partial x}(x,\xi)\right)(W_x) + \mu_\xi\left(\tfrac{\partial\tilde{f}}{\partial\xi}(x,\xi)\right) = -\mu\left((TL_{x^{-1}})_x(W_x)\right) +$$
$$+\mu_\xi\left((TL_{x^{-1}})_x(X)\right) + \xi([(TL_{x^{-1}})_x(X),(TL_{x^{-1}})_x(W_x)])$$

for arbitrary W_x and μ_ξ. As a consequence, we have $\frac{\partial\tilde{f}}{\partial\xi}(x,\xi) = (TL_{x^{-1}})_x(X)$ and $\frac{\partial\tilde{f}}{\partial x}(x,\xi) = \left(-\mu - ad^*_{(TL_{x^{-1}})_x(X)}\xi\right)\circ(TL_{x^{-1}})_x$. So, we get the result. ∎

Corollary 1. *The motions of the Hamiltonian system* $\left(G \times \mathcal{G}^*, \tilde{\omega}, \tilde{f}\right)$ *are described by the following differential equations*

$$\left\{\begin{aligned} \dot{x} &= (TL_x)_e \left(\frac{\partial \tilde{f}}{\partial \xi}(x,\xi)\right) \\ \dot{\xi} &= -(TL_x)_e^* \left(\frac{\partial \tilde{f}}{\partial x}(x,\xi)\right) - ad^*_{\frac{\partial \tilde{f}}{\partial \xi}(x,\xi)}\xi \end{aligned}\right. .$$

4.2. The tangent group of G.

4.2.1. *Basic notations.* The tangent group $G_1 = TG$ of a Lie group G is also a Lie group with group operations corresponding to the original ones of G by tangent prolongations (see [13] for more details). We shall denote the elements of G_1 by x_1 or g_1, where $\pi_{_G}(x_1) = x$ and $\pi_{_G}(g_1) = g$ for $\pi_{_G} : TG \to G$ the canonical projection. The identity of G_1 is denoted by e_1. So, the operations are

$$\begin{array}{ccc} G_1 \times G_1 & \longrightarrow & G_1 \\ (x_1, g_1) & \longmapsto & x_1 \cdot g_1 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} G_1 & \longrightarrow & G_1 \\ x_1 & \longmapsto & x_1^{-1} \end{array}$$

defined by

$$x_1 \cdot g_1 = (TR_g)_x(x_1) + (TL_x)_g(g_1) \ \in T_{xg}G$$

and

$$x_1^{-1} = -(TL_{x^{-1}})_e \circ (TR_{x^{-1}})_x(x_1) \in T_{x^{-1}}G.$$

From now on G_1 will represent the Lie group TG and $\mathcal{G}_1$ its Lie algebra. The left trivialization of the tangent bundle of G is now denoted by λ_{G_1}.

4.2.2. *Left trivializations.*

Lemma 2. [13] *By means of the left trivialization* λ_{G_1}*, we have* $\boxed{G_1 \simeq G \times \mathcal{G}}$ *and the group structure on* G_1 *looks as follows*

$$(x, Y) \cdot (g, Z) = (xg, Ad_{g^{-1}}Y + Z)$$

$$(x, Y)^{-1} = (x^{-1}, -Ad_x Y),$$

with $(x, Y), (g, Z) \in G \times \mathcal{G}$.

As a consequence of the previous Lemma, the Lie algebra $\mathcal{G}_1$ is isomorphic to $\mathcal{G} \times \mathcal{G}$, where the Lie product of two elements $(Y, Z), (Y', Z') \in \mathcal{G} \times \mathcal{G}$ is given by

$$[(Y, Z), (Y', Z')] = ([Y, Y'], [Y, Z'] - [Z, Y']). \tag{11}$$

Moreover, $\mathcal{G}_1^* \simeq \mathcal{G}^* \times \mathcal{G}^*$.

Since G_1 is a Lie group, we can consider the left trivialization of the tangent bundle TG_1, that is,

$$\begin{array}{rccc} \lambda_{TG_1}: & G_1 \times \mathcal{G}_1 & \longrightarrow & TG_1 \\ & (x_1, Y_1) & \longmapsto & \big(x_1, (TL_{x_1})_{e_1}(Y_1)\big) \end{array}.$$

This trivialization gives the isomorphism $TG_1 \simeq G_1 \times \mathcal{G}_1$. However, as we have seen, $G_1 \simeq G \times \mathcal{G}$ and $\mathcal{G}_1 \simeq \mathcal{G} \times \mathcal{G}$, by the left trivialization of G_1. Then, $TG_1 \simeq G \times \mathcal{G} \times \mathcal{G} \times \mathcal{G}$. In particular, $\boxed{T^2G \simeq G \times \mathcal{G} \times \mathcal{G}}$. We can also consider the left trivialization of T^*G_1,

$$\begin{array}{rccc} \lambda_{T^*G_1}: & G_1 \times \mathcal{G}_1^* & \longrightarrow & T^*G_1 \\ & (x_1, \xi_1) & \longmapsto & (x_1, (TL_{x_1^{-1}})^*_{x_1}(\xi_1)) \end{array},$$

to get $T^*G_1 \simeq G_1 \times \mathcal{G}_1^*$. Now, since $G_1 \simeq G \times \mathcal{G}$ and $\mathcal{G}_1^* \simeq \mathcal{G}^* \times \mathcal{G}^*$, we are able to conclude that $\boxed{T^*G_1 \simeq G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^*}$.

The cotangent bundle of G_1 will play an important rule in the next subsection, so we denote by λ the isomorphism from $G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^*$ to T^*G_1.

Remark 2. *By the above properties we can introduce now the local coordinates used in the rest of the paper. First introduce suitable coordinates for points in G choosing the type 1 coordinates (x^i) for a point $x \in G$. That is, the coordinates are such that $x = \exp\left(\sum_{i=1}^n x^i A_i\right)$, where $\exp : \mathcal{G} \to G$ is the exponential map and $\{A_i\}$ is the basis considered in subsection 4.1.3 for the Lie algebra $\mathcal{G}$. We can provide local coordinates for all the manifolds we shall be considering later:*

- *the coordinates for a point in $G_1 \simeq G \times \mathcal{G}$ will be represented by (x^i, y^j), where x^i represent the base point and y^j represent the coordinates of the element of the Lie algebra with respect to its base;*
- *analogously, the coordinates for $TG_1 \simeq G_1 \times \mathcal{G}_1 \simeq G \times \mathcal{G} \times \mathcal{G} \times \mathcal{G}$ will be (x^i, y^j, v^k, u^l) where the last two elements are the coordinates of the points in $\mathcal{G}$ with respect to the Lie algebra basis;*
- *in particular, the coordinates for a point in $T^2G \simeq G \times \mathcal{G} \times \mathcal{G}$ will be (x^i, y^j, u^l) where again y^j are the coordinates for an element of the*

Lie algebra and now u^k *are the coordinates for the new Lie algebra element;*

- *the coordinates for the manifold* $T^*G_1 \simeq G_1 \times \mathcal{G}_1^* \simeq G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^*$ *will be* (x^i, y^j, p_k, q_l) *where the last two elements are the coordinates of the points in* $\mathcal{G}^*$ *with respect to the dual basis of* $\{A_i\}$.

4.2.3. *Tangent map of the left translation.* The following result is valid when G is a matrix Lie group, from now on our group is considered as being of this type.

Proposition 2. *Let* $(x,Y),(g,Z) \in G\times\mathcal{G}$, *the tangent map of the left translation by* (x,Y) *at* (g,Z), *can be interpreted as*

$$\begin{array}{rrcl} (TL_{(x,Y)})_{(g,Z)}: & T_gG\times\mathcal{G} & \longrightarrow & T_{xg}G\times\mathcal{G} \\ & (W_g,W) & \longmapsto & (TL_{(x,Y)})_{(g,Z)}(W_g,W) \end{array}$$

with

$$(TL_{(x,Y)})_{(g,Z)}(W_g,W) = (\ (TL_x)_g(W_g)\ ,\ W + [Ad_{g^{-1}}Y, (TL_{g^{-1}})_g(W_g)]\ ).$$

Proof: The tangent map of $L_{(x,Y)}$ at (g,Z) maps elements of $T_{(g,Z)}(G\times\mathcal{G}) \simeq T_gG\times\mathcal{G}$ into elements of $T_{(xg,Ad_{g^{-1}}Y+Z)}(G\times\mathcal{G}) \simeq T_{xg}G\times\mathcal{G}$. So, given $w \in T_{(g,Z)}(G\times\mathcal{G})$, there is an isomorphism which allows us to identify $TL_{(x,Y)}{}_{(g,Z)}(w)$ with the following element of $T_{xg}G\times\mathcal{G}$:

$$\left((T\tau_{G})_{(xg,Ad_{g^{-1}}Y+Z)}(TL_{(x,Y)})_{(g,Z)}(w)\,,\,(T\tau_{\mathcal{G}})_{(xg,Ad_{g^{-1}}Y+Z)}(TL_{(x,Y)})_{(g,Z)}(w)\right),$$

where $\tau_G : G\times\mathcal{G}\to G$ and $\tau_{\mathcal{G}} : G\times\mathcal{G}\to\mathcal{G}$ are the canonical projections. This element can be view as

$$(\ T(\tau_G\circ L_{(x,Y)})_{(g,Z)}(w)\ ,\ T(\tau_{\mathcal{G}}\circ L_{(x,Y)})_{(g,Z)}(w)\ ).$$

Now considering $(W_g,W)\in T_gG\times\mathcal{G}$ such that $(W_g,W)\sim w\in T_{(g,Z)}(G\times\mathcal{G})$ and applying the Leibnitz formula, we obtain the required result. Indeed, fix (g,Z) and apply the Leibnitz formula to $f=\tau_G\circ L_{(x,Y)}$ and to $f'=\tau_{\mathcal{G}}\circ L_{(x,Y)}$ to develop the first and second coordinate of the latter element, respectively. In the first case and because $f(g,Z)=xg$, we get

$$Tf_{(g,Z)}(w) = (Tf_Z)_g(W_g) + (Tf_g)_Z(W) = (TL_x)_g(W_g),$$

since $f_Z = L_x$ and $(Tf_g)_Z(W) = 0$. In the second one, the map f' is defined by $f'(g, Z) = Ad_{g^{-1}}Y + Z$. Consequently,

$$T(\tau_{\mathcal{G}} \circ L_{(x,Y)})_{(g,Z)}(w) = (Tf'_g)_Z(W) + (Tf'_Z)_g(W_g),$$

with $(Tf'_g)_Z(W) = W$ and $(Tf'_Z)_g(W_g) = \frac{d}{dt}\left[Ad_{exp(-tg^{-1}W_g)}(Ad_{g^{-1}}Y)\right]_{|t=0}$. This last expression is reduce to $[Ad_{g^{-1}}Y, g^{-1}W_g]$. The result follows because on matrix Lie groups we have $g^{-1}W_g = (TL_{g^{-1}})_g(W_g)$. ■

Remark 3. *In particular,* $(TL_{(x,Y)})_{(e,0)} : \mathcal{G} \times \mathcal{G} \to T_xG \times \mathcal{G}$ *is such that*

$$(TL_{(x,Y)})_{(e,0)}(Z, W) = (\,(TL_x)_e(Z)\,,\, W + ad_Y Z\,).$$

Moreover, the transpose map $(TL_{(x,Y)})^*_{(e,0)} : (T_xG \times \mathcal{G})^* \to (\mathcal{G} \times \mathcal{G})^*$ *is such that, for each* $\mu \in (T_xG \times \mathcal{G})^*$ *we have*

$$\left[(TL_{(x,Y)})^*_{(e,0)}(\mu)\right](Z, W) = [(TL_x)^*_e(\xi_x)](Z) + \xi_Y(W + ad_Y Z),$$

where $\mu \sim (\xi_x, \xi_Y) \in T^*_xG \times \mathcal{G}^*$.

4.3. Riemannian cubic polynomials - variational approach. Consider the constructed orthogonal frame field A_i of G. Let V be the velocity vector field of a curve x given by $V = \sum_{i=1}^n y^i A_i(x)$ where $y^i, i = 1, \ldots, n$, are smooth functions of time. We have omitted the dependence in time to simplify the notation. Denote $\sum_{i=1}^n \dot{y}^i A_i(x)$ by $\dot{V}$ and, consequently, $\ddot{V}$ and $\dddot{V}$ are $\sum_{i=1}^n \ddot{y}^i A_i(x)$ and $\sum_{i=1}^n \dddot{y}^i A_i(x)$, respectively.

Theorem 3. [10] *A smooth curve* $x : [0, T] \to G$ *is a Riemannian cubic polynomial if and only if its velocity vector field* V *satisfies*

$$\dddot{V} + [V, \ddot{V}] = 0. \tag{12}$$

The equations (12) are the Euler-Lagrange equations (1) that define the RCP, but now specialize to the Lie group case. This result is proved in [10, 17], by using the fact that

$$\frac{DV}{dt} = \dot{V}, \qquad \frac{D^2V}{dt^2} = \ddot{V} + \frac{1}{2}\,[V, \dot{V}], \qquad \frac{D^3V}{dt^3} = \dddot{V} + [V, \ddot{V}] + \frac{1}{4}\,[V, [V, \dot{V}]]$$

and the properties (8) and (9). In this context, the two invariants along the cubic polynomial, I_1 and I_2 from proposition 1, become

$$I_1 = \frac{1}{2}\,\left\langle \dot{V}, \dot{V} \right\rangle - \left\langle \ddot{V}, V \right\rangle$$

and

$$I_2 = \left\langle \ddot{V}, \ddot{V} \right\rangle + \frac{1}{2} \left\langle [V, \dot{V}], 2\ddot{V} + [V, \dot{V}] \right\rangle .$$

4.4. Optimal control problem - presymplectic approach. The presymplectic description of the RCP's control problem presented in section 3 is now adapted to the compact Lie group case.

4.4.1. *Optimal control problem.* Consider the RCP's optimal control problem on G. The state space is G_1 and the control bundle T^2G. By using left trivializations of subsection 4.2.2, the optimal control problem consists in finding the C^2 piecewise smooth curve $\gamma = (x, Y, U) : [0, T] \to G \times \mathcal{G} \times \mathcal{G}$ with fixed endpoints in state space $G \times \mathcal{G}$, satisfying the control system

$$\frac{D^2 x}{dt^2} = (TL_x)_e(U)$$

and minimizing the functional $\int_0^T L(\gamma(t))dt$, where $T \in \mathcal{R}^+$ is fixed and the cost function $L : G \times \mathcal{G} \times \mathcal{G} \to \mathcal{R}$ is defined by

$$L(x, Y, U) = \frac{1}{2}\langle U, U\rangle.$$

Since we have an orthogonal frame field for G, the components of the metric are such that $g_{ij} = \delta_{ij}$, $i, j = 1, ..., n$. Then, the cost function is defined in local coordinates by $L(x^i, y^i, u^i) = \frac{1}{2}\sum_{i=1}^n (u^i)^2$. On the other hand, within the formalism of subsection 2.2, the control system is (2) where now

$$\begin{tikzcd}
G \times \mathcal{G} \times \mathcal{G} \arrow[r, "\Pi"] \arrow[dr, "\tau_2^1"'] & G \times \mathcal{G} \times \mathcal{G} \times \mathcal{G} \arrow[d, "\pi_{G\times\mathcal{G}}"] \\
& G \times \mathcal{G}
\end{tikzcd}$$

with τ_2^1 and $\pi_{G\times\mathcal{G}}$ being the natural projections onto the first two factors. As the connection corresponding to the bi-invariant metric on G is such that (7) holds, the vector field Π is given locally by $\Pi(x^i, y^i, u^i) = (x^i, y^i, y^i, u^i)$, expression that we can interpret as

$$\Pi(x, Y, U) = (x, Y, Y, U).$$

4.4.2. *Dynamics of the control system.* We introduce now the Hamiltonian description of our system having T^*G_1 as the co-state space. This space can be seen as being $G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^*$, once again by using the left trivializations. Thus, the total space of the presymplectic system $(\mathcal{T}, \Omega, H)$ is reduced to the vector bundle $\mathcal{T} = G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^* \times \mathcal{G}$. The elements of this space are represented by (x, Y, ξ, ξ_Y, U). Moreover, the images through the canonical projections of $\mathcal{T}$ into the co-state space and into the control bundle are, respectively, $pr_1(x, Y, \xi, \xi_Y, U) = (x, Y, \xi, \xi_Y)$ and $pr_2(x, Y, \xi, \xi_Y, U) = (x, Y, U)$.

The presymplectic two-form Ω on $\mathcal{T}$ is similar to (3). We define it as being the following pull-back of the canonical symplectic two form ω_0 on T^*G_1:

$$\Omega = (\lambda \circ pr_1)_* \omega_0.$$

The Hamiltonian function $H : \mathcal{T} \to \mathcal{R}$ is (4) which now becomes

$$H(x, Y, \xi, \xi_Y, U) = \xi(Y) + \xi_Y(U) - \frac{1}{2}\langle U, U\rangle,$$

by using the expressions obtained for Π and L in the previous subsection. Locally,

$$H(x^i, y^i, p_i, q_i, u^i) = \sum_{i=1}^{3} p_i y^i + \sum_{i=1}^{3} q_i u^i - \frac{1}{2}\sum_{i=1}^{3}(u^i)^2.$$

Since we have a regular optimal control problem, the solutions of the optimal control problem will lie in the symplectic manifold (W_1, Ω_{W_1}). As we have seen, in the RCP problem the constraint manifold W_1 is locally defined by $\frac{\partial H}{\partial u^i} = 0$, which, in the present case, gives $u^i = q_i$, $i = 1, ..., n$. Moreover, we can interpret this as follows.

$$W_1 = \{(x, Y, \xi, \xi_Y, U) \in \mathcal{T} \ : \ U = X_{\xi_Y}\},$$

where X_{ξ_Y} is the tangent vector identified with the co-vector ξ_Y. Then, W_1 is a submanifold of $\mathcal{T}$ diffeomorphic to $G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^*$. In that sense, the Hamiltonian on W_1 is defined by

$$H_{|W_1}(x, Y, \xi, \xi_Y) = \xi(Y) + \frac{1}{2}\xi_Y(X_{\xi_Y}) \tag{13}$$

and the two-forms Ω_{W_1} and ω_0 have the same local expression. Indeed, Ω_{W_1} can be identified with $\lambda_* \omega_0$.

We now proceed to get the Hamiltonian vector field Γ_{W_1} defined by the dynamical system (6). In order to apply the classic results from subsection 4.1.4 to the present higher order situation, we use the left trivializations of subsection 4.2.2 to realize Γ_{W_1} as a vector field on $G_1 \times \mathcal{G}_1^*$. So, let $\tilde{\Gamma}$ be the vector field on $G_1 \times \mathcal{G}_1^*$ such that $\Gamma_{W_1} \circ \lambda^{-1} \circ \lambda_{_{T^*G_1}} = T\left(\lambda^{-1} \circ \lambda_{_{T^*G_1}}\right) \circ \tilde{\Gamma}$. That is, $\tilde{\Gamma}$ is such that the following diagram is commutative:

$$\begin{tikzcd}
G_1 \times \mathcal{G}_1^* \arrow[r, "\tilde{\Gamma}"] \arrow[d, "\lambda^{-1}\circ\lambda_{_{T^*G_1}}"'] & T(G_1 \times \mathcal{G}_1^*) \arrow[d, "T(\lambda^{-1}\circ\lambda_{_{T^*G_1}})"] \\
G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^* \arrow[r, "\Gamma_{W_1}"'] & T(G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^*)
\end{tikzcd}$$

This is the Hamiltonian vector field defined by the dynamical system

$$i_{\tilde{\Gamma}}\tilde{\Omega} = d\tilde{H}, \tag{14}$$

where $\tilde{H}$ and $\tilde{\Omega}$ are, respectively, the Hamiltonian function and the symplectic form realized on $G_1 \times \mathcal{G}_1^*$. That is, $\tilde{H} = H_{|W_1} \circ \lambda^{-1} \circ \lambda_{_{T^*G_1}}$ and $\tilde{\Omega} = \left(\lambda^{-1} \circ \lambda_{_{T^*G_1}}\right)_* (\Omega_{W_1})$. In that sense, the dynamical systems (6) and (14) are equivalent.

Note that, by the identification of Ω_{W_1} with $\lambda_*\omega_0$, the two form $\tilde{\Omega}$ can be seen as being $\left(\lambda^{-1} \circ \lambda_{_{T^*G_1}}\right)_* (\lambda_*\omega_0)$. But this is just $\tilde{\omega}_0 = \left(\lambda_{_{T^*G_1}}\right)_* (\omega_0)$ the left trivialization of the canonical two-form ω_0.

We use corollary 1 to conclude that each integral curve $(x_1(t), \xi_1(t))$ of $\tilde{\Gamma}$ satisfies the following differential equations:

$$\left\{\begin{array}{l}
\dot{x}_1 = (TL_{x_1})_{e_{G_1}}\left(\dfrac{\partial \tilde{H}}{\partial \xi_1}(x_1, \xi_1)\right) \\
\dot{\xi}_1 = -(TL_{x_1})^*_{e_{G_1}}\left(\dfrac{\partial \tilde{H}}{\partial x_1}(x_1, \xi_1)\right) - ad^*_{\frac{\partial \tilde{H}}{\partial \xi_1}(x_1,\xi_1)}\xi_1
\end{array}\right. . \tag{15}$$

The interpretation of the above differential equations allows us to describe the dynamical system (6) in the present case. In order to simplify the notation, in what follows, Γ denotes Γ_{W_1} and H denotes H_{W_1}.

Theorem 4. *Consider an integral curve $(x(t), Y(t), \xi(t), \xi_Y(t))$ of Γ. Then, the following set of equations is satisfied:*

$$\begin{cases} \dot{x} = (TL_x)_e \left(\frac{\partial H}{\partial \xi}(z)\right) \\ \dot{Y} = \frac{\partial H}{\partial \xi_Y}(z) + ad_Y \frac{\partial H}{\partial \xi}(z) \\ \dot{\xi} = -\frac{\partial H}{\partial x}(z) \circ (TL_x)_e + ad_Y^* \frac{\partial H}{\partial Y}(z) - ad^*_{\frac{\partial H}{\partial \xi}(z)} \xi - ad^*_{\frac{\partial H}{\partial \xi_Y}(z)} \xi_Y \\ \dot{\xi}_Y = -\frac{\partial H}{\partial Y}(z) - ad^*_{\frac{\partial H}{\partial \xi}(z)} \xi_Y \end{cases}.$$

where z denotes the element (x, Y, ξ, ξ_Y).

Proof: Let $(x_1(t), \xi_1(t)) \in G_1 \times \mathcal{G}_1^*$ be the integral curve of $\tilde{\Gamma}$ corresponding to the integral curve $(x(t), Y(t), \xi(t), \xi_Y(t))$ by left trivialization. Note that, we can identify the elements $\dot{x}_1 \in T_{x_1}G_1$ and $\frac{\partial \tilde{H}}{\partial \xi_1}(z) \in \mathcal{G}_1$ with new elements $\dot{x}_1 \sim (\dot{x}, \dot{Y}) \in T_xG \times \mathcal{G}$ and $\frac{\partial \tilde{H}}{\partial \xi_1}(z) \sim \left(\frac{\partial H}{\partial \xi}(z), \frac{\partial H}{\partial \xi_Y}(z)\right) \in \mathcal{G} \times \mathcal{G}$, respectively. Moreover, we have

$$(TL_{x_1})_{e_{G_1}} \left(\frac{\partial \tilde{H}}{\partial \xi_1}(z)\right) \sim \left(TL_{(x,Y)}\right)_{(e,0)} \left(\frac{\partial H}{\partial \xi}(z), \frac{\partial H}{\partial \xi_Y}(z)\right).$$

Now by remark 3, we develop the right element of this to get

$$(TL_{x_1})_{e_{G_1}} \left(\frac{\partial \tilde{H}}{\partial \xi_1}(z)\right) \sim \left((TL_x)_e \left(\frac{\partial H}{\partial \xi}(z)\right) \ , \ \frac{\partial H}{\partial \xi_Y}(z) + ad_Y \frac{\partial H}{\partial \xi}(z)\right).$$

Consequently, the first equation of (15) is interpreted as

$$\begin{cases} \dot{x} = (TL_x)_e \left(\frac{\partial H}{\partial \xi}(z)\right) \\ \dot{Y} = \frac{\partial H}{\partial \xi_Y}(z) + ad_Y \frac{\partial H}{\partial \xi}(z) \end{cases}.$$

We now proceed to get the other two required equations. Identify the element $\frac{\partial \tilde{H}}{\partial x_1}(z) \in T^*_{x_1}G_1$ with a new element $\left(\frac{\partial H}{\partial x}(z), \frac{\partial H}{\partial Y}(z)\right) \in T^*_xG \times \mathcal{G}^*$. Consider the following identifications:

$$(TL_{x_1})^*_{e_{G_1}} \left(\frac{\partial \tilde{H}}{\partial x_1}(z)\right) \sim \left(TL_{(x,Y)}\right)^*_{(e,0)} \left(\frac{\partial H}{\partial x}(z), \frac{\partial H}{\partial Y}(z)\right)$$

and

$$ad^*_{\frac{\partial \tilde{H}}{\partial \xi_1}(x_1,\xi_1)} \xi_1 \sim ad^*_{\left(\frac{\partial H}{\partial \xi}(z), \frac{\partial H}{\partial \xi_Y}(z)\right)} (\xi, \xi_Y).$$

Now use remark 3 to obtain

$$\begin{aligned}&\left(TL_{(x,Y)}\right)^*_{(e,0)}\left(\tfrac{\partial H}{\partial x}(z),\tfrac{\partial H}{\partial Y}(z)\right)(Z,W)=\\&=\left[\tfrac{\partial H}{\partial x}(z)\circ\left(TL_x\right)_e\right](Z)+\left(\tfrac{\partial H}{\partial Y}(z)\right)(W+ad_YZ)\end{aligned}$$

and the algebra structure (11), to write

$$\begin{aligned}&ad^*_{\left(\frac{\partial H}{\partial \xi}(z),\frac{\partial H}{\partial \xi_Y}(z)\right)}(\xi,\xi_Y)(Z,W)=\\&=-\xi\circ ad_{\frac{\partial H}{\partial \xi}(z)}Z-\xi_Y\circ ad_{\frac{\partial H}{\partial \xi}(z)}W-\xi_Y\circ ad_{\frac{\partial H}{\partial \xi_Y}(z)}Z\end{aligned}$$

for arbitrary $(Z,W)\in\mathcal{G}\times\mathcal{G}$. So, the second equation of (15) allows us to write

$$\begin{aligned}&\dot{\xi}(Z)+\dot{\xi}_Y(W)=\xi\circ ad_{\frac{\partial H}{\partial \xi}}Z+\xi_Y\circ ad_{\frac{\partial H}{\partial \xi}(z)}W+\xi_Y\circ ad_{\frac{\partial H}{\partial \xi_Y}(z)}Z-\\&-\left[\tfrac{\partial H}{\partial x}(z)\circ\left(TL_x\right)_e\right](Z)-\left(\tfrac{\partial H}{\partial Y}(z)\right)(W+ad_YZ)\end{aligned}$$

and the result follows. ■

Corollary 2. *Considering the RCP Hamiltonian defined by (13), the Hamiltonian equations from theorem 4 are reduced to*

$$\begin{cases}\dot{x}=\left(TL_x\right)_eY\\ \dot{Y}=X_{\xi_Y}\\ \dot{\xi}=0\\ \dot{\xi}_Y=-\xi-ad^*_Y\xi_Y\end{cases}.$$

Proof: According the expression of our Hamiltonian function (13), we have $\frac{\partial H}{\partial x}(z)=0$, $\frac{\partial H}{\partial Y}(z)=\xi$, $\frac{\partial H}{\partial \xi}(z)=Y$ and $\frac{\partial H}{\partial \xi_Y}(z)=X_{\xi_Y}$. Now substitute these expressions in the equations from theorem 4 to obtain

$$\begin{cases}\dot{x}=\left(TL_x\right)_e(Y)\\ \dot{Y}=X_{\xi_Y}\\ \dot{\xi}=-ad^*_{X_{\xi_Y}}\xi_Y\\ \dot{\xi}_Y=-\xi-ad^*_Y\xi_Y\end{cases}.$$

But, for $Z\in\mathcal{G}$, $\left(-ad^*_{X_{\xi_Y}}\xi_Y\right)(Z)=\left(\xi_Y\circ ad_{X_{\xi_Y}}\right)(Z)=<X_{\xi_Y},[X_{\xi_Y},Z]>$. The last one is equal to zero, by property (9). Then, the result follows. ■

4.5. Relation between the two approaches. We now establish the relation between the current Hamiltonian approach from the previous subsection and the variational one presented in subsection 4.3. Let V, $\dot{V}$, $\ddot{V}$ and $\dddot{V}$ be the elements introduced in the afore mentioned subsection. Recall the notation assumed in subsection 4.1.3.

Lemma 3. *Along the optimal control problem solution curve, we have*

$$\begin{aligned} V &= (TL_x)_e\,(Y) \\ \dot{V} &= (TL_x)_e\,(X_{\xi_Y}) \\ \ddot{V} &= (TL_x)_e(-X_\xi - ad_Y X_{\xi_Y}) \\ \dddot{V} &= (TL_x)_e\,(ad_Y X_\xi + ad_Y ad_Y X_{\xi_Y})\,. \end{aligned}$$

Proof: Use corollary 2 to prove the result. The expression of V comes directly from the first equation of the corollary since $V = \dot{x}$. Consequently, $\dot{V} = (TL_x)_e(\dot{Y})$ which by the second equation of the mentioned corollary gives $\dot{V} = (TL_x)_e\,(X_{\xi_Y})$. Furthermore, using the last equation of the corollary, we get $\ddot{V} = (TL_x)_e(\dot{X}_{\xi_Y}) = (TL_x)_e(X_{\dot{\xi}_Y}) = (TL_x)_e(-X_\xi - X_{ad^*_Y \xi_Y})$ and the result follows. Finally, to obtain the expression of $\dddot{V}$ write it as $\dddot{V} = (TL_x)_e\left(-\dot{X}_\xi - \dot{X}_{ad^*_Y\xi_Y}\right)$. Now, observe that $\dot{X}_\xi = X_{\dot{\xi}} = 0$. On the other hand, $\dot{X}_{ad^*_Y\xi_Y} = X_{ad^*_Y\dot{\xi}_Y} + X_{ad^*_{\dot{Y}}\xi_Y} = -X_{ad^*_Y\xi} - X_{ad^*_Y ad^*_Y \xi_Y} + X_{ad^*_{X_{\xi_Y}}\xi_Y}$. But $ad^*_{X_{\xi_Y}}\xi_Y$ vanishes by property (9) and the proof is complete. ∎

Proposition 3. *The Hamiltonian equations from corollary 2 imply the Euler-Lagrange equations (12).*

Proof: From lemma 3 we see that $(TL_{x^{-1}})_x(\dddot{V} + [V, \ddot{V}]) = 0$. ∎

According corollary 2, the solution dynamical vector field Γ is defined at each $z = (x, Y, \xi, \xi_Y) \in G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^*$ by

$$\Gamma(z) = ((TL_x)_e\,(Y)\,, X_{\xi_Y}, 0, -\xi - ad^*_Y\xi_Y).$$

In order to prove that a function I defined in $G \times \mathcal{G} \times \mathcal{G}^* \times \mathcal{G}^*$ is a conserved quantity, using the Hamiltonian formalism, it is enough to prove that $\Gamma(I) = 0$. To develop this, we adopt the following notation, similar to the one considered in subsection 4.1.4:

$$(dI)(z)(\Gamma(z)) = \left(\frac{\partial I}{\partial x}(z)\right)\left((TL_x)_e(Y)\right) + \left(\frac{\partial I}{\partial Y}(z)\right)(X_{\xi_Y}) -$$
$$-\xi\left(\frac{\partial I}{\partial \xi_Y}(z)\right) - ad_Y^*\xi_Y\left(\frac{\partial I}{\partial \xi_Y}(z)\right).$$

Proposition 4. *The following two expressions are invariants along the extremal trajectories of the RCP's optimal control problem:*

$$\hat{I}_1 = \frac{1}{2}\xi_Y(X_{\xi_Y}) + \xi(Y)$$

and

$$\hat{I}_2 = \xi\left(X_\xi\right) + \left(ad_Y^*\xi_Y\right)\left(X_\xi\right) + \frac{1}{2}\left(ad_Y^* ad_Y^*\xi_Y\right)\left(X_{\xi_Y}\right).$$

Proof: Deduce that $\frac{\partial \hat{I}_1}{\partial x}(z) = \frac{\partial \hat{I}_2}{\partial x}(z) = 0$, $\frac{\partial \hat{I}_1}{\partial Y}(z) = \xi$, $\frac{\partial \hat{I}_1}{\partial \xi_Y}(z) = X_{\xi_Y}$, $\frac{\partial \hat{I}_2}{\partial Y}(z) = \left\langle \,.\,, ad_{X_{\xi_Y}}\left(X_\xi + ad_Y X_{\xi_Y}\right)\right\rangle$ and $\frac{\partial \hat{I}_2}{\partial \xi_Y}(z) = -ad_Y X_\xi - ad_Y ad_Y X_{\xi_Y}$. Now is simply to prove that $(d\hat{I}_1)(\Gamma) = 0$ and $(d\hat{I}_2)(\Gamma) = 0$. ■

The first invariant coincides with the Hamiltonian function (13), so these two invariants are in involution.

To finish observe that the invariants from the above proposition are precisely the invariants I_1 and I_2 from the variational formalism presented in subsection 4.3. Indeed, use lemma 3 and the bi-invariance of the metric to write $I_1 = \frac{1}{2}\left\langle X_{\xi_y}, X_{\xi_y}\right\rangle + \left\langle X_\xi, Y\right\rangle + \left\langle ad_Y X_{\xi_Y}, Y\right\rangle$. Now, by property (9), we have $\left\langle ad_Y X_{\xi_Y}, Y\right\rangle = -\left\langle X_{\xi_Y}, ad_Y Y\right\rangle = 0$. Then, $I_1 = \hat{I}_1$. On the other hand, once again by lemma 3 and the invariance of the metric we are able to deduce that $\frac{1}{2}\left\langle [V,\dot{V}], 2\ddot{V} + [V,\dot{V}]\right\rangle = -\left\langle X_\xi, ad_Y X_{\xi_Y}\right\rangle - \frac{1}{2}\left\langle ad_Y X_{\xi_Y}, ad_Y X_{\xi_Y}\right\rangle$ and $\left\langle \ddot{V}, \ddot{V}\right\rangle = \left\langle X_\xi, X_\xi + 2ad_Y X_{\xi_Y}\right\rangle + \left\langle ad_Y X_{\xi_Y}, ad_Y X_{\xi_Y}\right\rangle$. Now substitute these expressions in I_2 to obtain

$$I_2 = \left\langle X_\xi, X_\xi + ad_Y X_{\xi_Y}\right\rangle + \frac{1}{2}\left\langle ad_Y X_{\xi_Y}, ad_Y X_{\xi_Y}\right\rangle = \hat{I}_2.$$

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