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DIAGONAL MINUS TAIL FORMS AND LASSERRE'S SUFFICIENT CONDITIONS FOR SUMS OF SQUARES

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ABSTRACT: Using our recent results on diagonal minus tail forms, we give an easily tested sufficient condition for a polynomial $f(\underline{x}) = \sum_{\underline{i}\in I} f_{\underline{i}}\underline{x}^{\underline{i}}$ in $\mathbb{R}[\underline{x}] = \mathbb{R}[x_1,\dots,x_n]$, to be a sum of squares of polynomials (sos). We show that the class of polynomials passing this test is wider than the class passing Lasserre's recent conditions. Another sufficient condition for f to be sos, like Lasserre's piecewise linear in the $f_{\underline{i}}$, is also given.

1. Introduction

In a recent paper [FK] we investigated homogeneous polynomials (or forms) $F(\underline{x}) \in \mathbb{R}[\underline{x}] = \mathbb{R}[x_1,\dots,x_n]$ of aspect $F(\underline{x}) = D(\underline{x}) - T(\underline{x})$, with a diagonal form $D(\underline{x}) = \sum_{i=1}^n b_i x_i^{2d}$ and a tail $T(\underline{x}) = \sum_{\underline{i}\in I} a_{\underline{i}}\underline{x}^{\underline{i}}$ in which all $\underline{i} = (i_1,\dots,i_n)$ have norm $|\underline{i}| = \sum_{\nu=1}^n i_\nu = 2d$, at least two nonzero entries, and all occurring $a_{\underline{i}} > 0$. We called these polynomials diagonal minus tail (dmt) forms. The main result of [FK] says that a dmt form is positive semidefinite (psd) if and only if it is a sum of squares of polynomials (sos).

From this we derived a sufficient condition for an arbitrary form to be sos. The aim of this paper is to strengthen this condition a little bit and to put it in the context of Lasserre's recent sufficient conditions [La] for a form to be sos.

In section 2 we define the class $\mathcal{C}_L$ of polynomials as that satisfying Lasserre's conditions and associate to a homogeneous polynomial f a certain dmt form, the hat of f, $\hat{f}$. We show in theorem 2.1 that if f belongs to the class called $\mathcal{C}^{\mathrm{hat}}_{\mathrm{psd}}$, i.e. if $\hat{f}$ is psd, then f is sos. As proposition 2.2 we find that if $f \in \mathcal{C}_L$ then any of its homogenizations is in $\mathcal{C}^{\mathrm{hat}}_{\mathrm{psd}}$. In this sense '$\mathcal{C}_L \subseteq \mathcal{C}^{\mathrm{hat}}_{\mathrm{psd}}$'. Finally we argue invoking results of [FK] that the decision of whether $\hat{f}$ (or any other dmt form) is psd comes down to a single application of a simple minimization procedure.

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We should mention that there exists an algorithm for deciding whether an arbitrary form is sos devised by Powers and Woermann [PW]. The criterion there given is recognizable as a problem rapidly solvable by semidefinite programming, see Parrilo [Pa], and the very useful survey by Laurent [Lau]. But semidefinite programming is a sophisticated nonclassical optimization method.

Lasserre emphasizes that till his work there was no simple sufficient condition for a polynomial to be sos expressed directly in terms of its coefficients. It is the aim of section 3 to join to Lasserre's another condition, piecewise linear in the coefficients of a polynomial, which guarantees it to be sos. Lasserre's proof uses the ties existing between the theory of moments and sos representations, again see [Lau]. He also uses results with Netzer [LaN] on the approximation of psd polynomials by sums of squares. We use very different ideas which most readers will consider more elementary.

Note: We discovered the connections of [FK] with [La] shortly after [FK] was submitted for prepublication. For submission to a journal, the present paper and [FK] will be merged into one.

2. The classes $\mathcal{C}_L$ and $\mathcal{C}_{\text{psd}}^{\text{hat}}$.

Let $\mathbb{R}[\underline{x}; \deg \leq 2d]$ and $\mathbb{R}[\underline{x}; \deg = 2d]$ denote the real vectorspaces of polynomials over $\mathbb{R}$ of degree$\leq 2d$ and of homogeneous polynomials over $\mathbb{R}$ of degree $2d$, respectively. For fixed d define the universe of n-tuples $\Omega = \{\underline{i} \in \mathbb{Z}^n_{\geq 0} : |\underline{i}| \leq 2d$ and at least two nonzero entries $\}$. Let $\Gamma = \{\underline{i} \in \Omega : \underline{i} \in 2\mathbb{Z}^n_{\geq 0}\}$, and let Γ^c be the complement of Γ in Ω.

Then $\Gamma = \{\underline{i} \in 2\mathbb{Z}^n_{\geq 0} : |\underline{i}| \leq 2d, \underline{i} \notin \{2d\underline{e}_1, \ldots, 2d\underline{e}_n\}\}$ is the set of even lattice points of 1-norm$\leq 2d$ that are not essential in the sense of Lasserre. Lasserre does not exclude the $2d\underline{e}_i$ in his definition of Γ but for our development it is convenient; and for the definition of the class $\mathcal{C}_L$ below it is easily seen to be irrelevant.

We define the class $\mathcal{C}_L$ of real polynomials $f(\underline{x}) = \sum_{\underline{i}} f_{\underline{i}} \underline{x}^{\underline{i}}$ as those satisfying the conditions of theorem 3 of [La]; that is: $f \in \mathbb{R}[\underline{x}; \deg \leq 2d]$ is in class $\mathcal{C}_L$ iff

$$\begin{aligned} f_{\underline{0}} - \sum_{\underline{i}\in\Gamma^c} |f_{\underline{i}}| + \sum_{\underline{i}\in\Gamma} \min\{0, f_{\underline{i}}\} &\geq 0, \\ \min_{i=1,\dots,n} f_{2d\underline{e}_i} - \sum_{\underline{i}\in\Gamma^c} |f_{\underline{i}}| \frac{|\underline{i}|}{2d} + \sum_{\underline{i}\in\Gamma} \min\{0, f_{\underline{i}}\} \frac{|\underline{i}|}{2d} &\geq 0. \end{aligned}$$

Note that a polynomial in $\mathcal{C}_L$ with $f_{\underline{0}} = 0$ or some $f_{2d\underline{e}_i} = 0$ is necessarily a nonnegative combination of monomials $\underline{x}^{\underline{i}}$ with even lattice points $\underline{i}$, i.e. $\underline{i} \in \Gamma \cup \{2d\underline{e}_1, ..., 2d\underline{e}_n\}$, and therefore obviously a sum of squares. So, when directly applied, homogeneous polynomials pass Lasserre's test only in cases that are clear by themselves. This fact does not mean that Lasserre's criterion is useless for homogeneous polynomials; for by an easy lemma, see e.g. [Lau, Lemma 3.3], a polynomial is sos iff its (de)homogenization is sos. Lasserre showed:

Theorem L. If $f \in \mathbb{R}[\underline{x}; \deg \leq 2d] \cap \mathcal{C}_L$, then f is sos.

At the other hand we can derive from our main result on dmt forms the following slight strengthening of corollary 2.9 of [FK].

Theorem 2.1 Let $f(\underline{x}) = \sum f_{\underline{i}} \underline{x}^{\underline{i}}$ be a (homogeneous) polynomial in $\mathbb{R}[\underline{x}; \deg = 2d]$, and define the *hat of f* by

$$\hat{f}(\underline{x}) = \sum_{i=1}^{n} f_{2d\underline{e}_i} x_i^{2d} - \sum_{\underline{i}\in\Gamma^c} |f_{\underline{i}}| \underline{x}^{\underline{i}} + \sum_{\underline{i}\in\Gamma} \min\{0, f_{\underline{i}}\} \underline{x}^{\underline{i}}.$$

If $\hat{f}$ is psd then f is sos.

Proof. For the purpose of this proof all $\underline{i}$ we refer can and will be supposed (to be in Ω) and to satisfy $|\underline{i}| = 2d$. With this understanding define

$$\Gamma_- = \{\underline{i} \in \Gamma : f_{\underline{i}} < 0\}, \quad \Gamma_+ = \{\underline{i} \in \Gamma : f_{\underline{i}} > 0\}.$$

Then, since f is homogeneous of degree $2d$, we can write

$$\hat{f}(\underline{x}) = \sum_{i=1}^{n} f_{2d\underline{e}_i} x_i^{2d} - \sum_{\underline{i}\in\Gamma^c\cup\Gamma_-} |f_{\underline{i}}| \underline{x}^{\underline{i}}.$$

This exhibits $\hat{f}$ as a dmt form. Assume now it is psd. We know by the proof of theorem 2.7 of [FK] that there are psd diagonal forms $D(\underline{x})$, and $D_{\underline{i}}(\underline{x})$ for each $\underline{i} \in \Gamma^c \cup \Gamma_-$, so that

$$E_{\underline{i}}(\underline{x}) = D_{\underline{i}}(\underline{x}) - |f_{\underline{i}}|\underline{x}^{\underline{i}} \text{ is psd, and } \hat{f} = D + \sum_{\underline{i} \in \Gamma^c \cup \Gamma_-} E_{\underline{i}}.$$

If f is indecomposable, so is $\hat{f}$ and all the $E_{\underline{i}}$ are as well. Then by theorem 2.3 of [FK] $E_i'(\underline{x}) = D_{\underline{i}}(\underline{x}) + f_{\underline{i}}\underline{x}^{\underline{i}}$ is sos. Note that the diagonal parts of $\hat{f}$ and f are equal. So this diagonal part is

$$\sum_{i=1}^{n} f_{2d\underline{e}_i} x_i^{2d} = D(\underline{x}) + \sum_{\underline{i} \in \Gamma^c \cup \Gamma_-} D_{\underline{i}}(\underline{x}).$$

This now yields

$$\begin{aligned} f(\underline{x}) &= \sum_{i=1}^{n} f_{2d\underline{e}_i} x_i^{2d} + \sum_{\underline{i} \in \Gamma^c \cup \Gamma_-} f_{\underline{i}}\underline{x}^{\underline{i}} + \sum_{\underline{i} \in \Gamma_+} f_{\underline{i}}\underline{x}^{\underline{i}} \\ &= D(\underline{x}) + \sum_{\underline{i} \in \Gamma^c \cup \Gamma_-} E_{\underline{i}}'(\underline{x}) + \sum_{\underline{i} \in \Gamma_+} f_{\underline{i}}\underline{x}^{\underline{i}}. \end{aligned}$$

By our observations on $D, E_{\underline{i}}'$, and since the definition of Γ_+ implies that the last summand is sos, $f(\underline{x})$ is sos. For decomposable forms, the proof can now be completed as in [FK]. $\square$

We noted that the hat of a homogeneous polynomial f is a diagonal minus tail form. It can be defined as the polynomial obtained by

- deleting all terms $f_{\underline{i}}x^{\underline{i}}$ for which $\underline{i} \in \Gamma_+$;
- changing all positive terms pertaining to non-even $\underline{i}$ to their negative counterpart.

So taking the hat is a closure operation: $\hat{\hat{f}} = \hat{f}$.

We define the class $\mathcal{C}_{\mathrm{psd}}^{\mathrm{hat}} = \{f : f \text{ is a form of even degree whose hat is psd}\}$.

As to the relation between the classes $\mathcal{C}_L$ and $\mathcal{C}_{\mathrm{psd}}^{\mathrm{hat}}$ we have the following.

Proposition 2.2. Let $f \in \mathbb{R}[\underline{x}; \deg \leq 2d] \cap \mathcal{C}_L$, and let $f_h \in \mathbb{R}[\underline{x}, x_{n+1}; \deg = 2d]$ be its homogenization. Then f_h lies in $\mathcal{C}_{\mathrm{psd}}^{\mathrm{hat}}$.

Proof. With Γ as in Lasserre's conditions, we can write

$$f(\underline{x}) = f_{\underline{0}} + \sum_{i=1}^{n} f_{2d\underline{e}_i} x_i^{2d} + \sum_{\underline{i}\in\Gamma^c} f_{\underline{i}} \underline{x}^{\underline{i}} + \sum_{\underline{i}\in\Gamma_-} f_{\underline{i}} \underline{x}^{\underline{i}} + \sum_{\underline{i}\in\Gamma_+} f_{\underline{i}} \underline{x}^{\underline{i}}.$$

Writing $f_{2d\underline{e}_{n+1}} := f_{\underline{0}}$, the homogenization of this polynomial is

$$f_h(\underline{x}, x_{n+1}) = \sum_{i=1}^{n+1} f_{2d\underline{e}_i} x_i^{2d} + \sum_{\underline{i}\in\Gamma^c} f_{\underline{i}} \underline{x}^{\underline{i}} x_{n+1}^{2d-|\underline{i}|} + \sum_{\underline{i}\in\Gamma_-} f_{\underline{i}} \underline{x}^{\underline{i}} x_{n+1}^{2d-|\underline{i}|} + \sum_{\underline{i}\in\Gamma_+} f_{\underline{i}} \underline{x}^{\underline{i}} x_{n+1}^{2d-|\underline{i}|}.$$

Now observe that $\underline{i} \in \mathbb{Z}^n$ is an even lattice point if and only if $(\underline{i}, 2d - |\underline{i}|) \in \mathbb{Z}^{n+1}$ is even. It follows that the hat of f_h is

$$\widehat{f_h}(\underline{x}, x_{n+1}) = \sum_{i=1}^{n+1} f_{2d\underline{e}_i} x_i^{2d} - \sum_{\underline{i}\in\Gamma^c} |f_{\underline{i}}| \underline{x}^{\underline{i}} x_{n+1}^{2d-|\underline{i}|} + \sum_{\underline{i}\in\Gamma} \min\{0, f_{\underline{i}}\} \underline{x}^{\underline{i}} x_{n+1}^{2d-|\underline{i}|}.$$

The dehomogenization of this polynomial is obtained by putting $x_{n+1} = 1$, yielding

$$\widehat{f_h}(\underline{x}, 1) = f_{\underline{0}} + \sum_{i=1}^{n} f_{2d\underline{e}_i} x_i^{2d} - \sum_{\underline{i}\in\Gamma^c} |f_{\underline{i}}| \underline{x}^{\underline{i}} + \sum_{\underline{i}\in\Gamma} \min\{0, f_{\underline{i}}\} \underline{x}^{\underline{i}}.$$

Recalling that the original polynomial is to satisfy Lasserre's conditions, we see that $\widehat{f_h}(\underline{x}, 1)$ satisfies Lasserre's conditions, simply because $|-|f_{\underline{i}}|| = |f_{\underline{i}}|$ and $\min\{0, \min\{0, f_{\underline{i}}\}\} = \min\{0, f_{\underline{i}}\}$.

Therefore $\widehat{f_h}(\underline{x}, 1)$ is sos, hence psd; and so $\widehat{f_h}(\underline{x}, x_{n+1})$ will be psd. Precisely this was to be proved. □

This result shows that every polynomial that passes Lasserre's condition for sos-ness, will after homogenization pass our test as well and therefore be sos. Note, however, that we have used Lasserre's own result in this proof and not invoked theorem 2.1, although the sharpness of theorem 2.1 leads one to believe that sos-ness of polynomials in $\mathcal{C}_L$ should be derivable from it.

Of course Lasserre's condition is very simple and our's would be of academic interest only, if it would be complicate to handle. But this is not the case. First note the dmt-ness of the hat of a polynomial assures that each of its indecomposable components is dmt: see [FK] lemma 2.1b. The indecomposable components of a dmt form and their dimensions are of course easy to find. Next, the proof of [FK] lemma 2.4 can be adapted to show that every local minimum of an indecomposable dmt form of dimension $n \geq 2$ and viewed as a function $f : \Delta_{n-1} \to \mathbb{R}$ must lie in $\operatorname{int}(\Delta_{n-1})$. This means that standard minimization methods to look for a local minimum of f on Δ_{n-1} will halt at a point in $\operatorname{int}(\Delta_{n-1})$. Finally, we have the following lemma.

Lemma 2.3. Let f be an indecomposable dmt form of dimension n. Then for every local minimum in $\underline{u} \in \operatorname{int}(\Delta_{n-1})$ there holds i or ii:
i. $f(\underline{u}) < 0$.
ii. $f(\underline{u}) \geq 0$ and $\underline{u}$ is global minimum for $f|\Delta_{n-1}$.
In case ii, f is psd.

Proof. Let $\underline{u}$ be a local minimum with $f(\underline{u}) \geq 0$. The proof of lemma 2.6 in [FK], does only depend on the fact that $f(\underline{u}) \geq 0$ (and not on the hypothesis enunciated there that f be psd). It gives us a Lagrange multiplier $\lambda \leq 0$. This multiplier together with that $\underline{u}$ be a local minimum of f such that $f(\underline{u}) \geq 0$ is used in the proof of theorem 2.7 in [FK] in order to represent f as a sum of elementary psd dmt forms. (Again, that proof does not make use of the full strength of the hypothesis that f be psd.) It follows that f is psd. Now corollary 2.8 of that paper gives that f has $\underline{u}$ as its global minimum. $\square$

By this lemma either the value of f at a local minimum in Δ_{n-1} is negative and f thus certainly not psd; or the value is nonnegative, the local minimum in fact global and thus we can be sure that the function is psd. So testing whether a homogeneous polynomial belongs to class $\mathcal{C}_{\text{psd}}^{\text{hat}}$ should be considered easy.

3. A piecewise linear condition for being sos

Here we use the criterion of theorem 2.1 to add to Lasserre's sufficient condition for sos-ness a further one which as his' is piecewise linear in the coefficients of f.

Lemma 3.1. The minimum and maximum values on Δ_{n-1} of a diagonal form $\sum_{i=1}^n b_i x_i^{2d}$ with all $b_i > 0$, and a term $x_1^{i_1} \cdots x_n^{i_n}$ with $|\underline{i}| = 2d$, are given by

$$\min\{\sum_{i=1}^n b_i x_i^{2d} : \underline{x} \in \Delta_{n-1}\} = \left(\sum_{i=1}^n b_i^{1/(1-2d)}\right)^{1-2d},$$

and

$$\max\{x_1^{i_1} \cdots x_n^{i_n} : \underline{x} \in \Delta_{n-1}\} = (2d)^{-2d} i_1^{i_1} \cdots i_n^{i_n}.$$

Proof. Put $F(\underline{x}) = \sum_{i=1}^n b_i x_i^{2d}$. Since $\mathrm{bd}(\Delta)$ is compact, and all $b_i > 0$, there exists a point $\hat{\underline{x}} \in \mathrm{bd}(\Delta)$ so that $0 < F(\hat{\underline{x}}) = \min F|\mathrm{bd}(\Delta)$. From now on one can reason as in the third part of the proof of lemma 2.4 of [FK] to see that the minimum of the function $\Delta_{n-1} \ni \underline{x} \mapsto F(\underline{x})$ is assumed in the interior of Δ_{n-1}. This justifies the use of the Lagrange multiplier method in a standard way to obtain the rest of the claim concerning F. We omit the details.

For the form $F(\underline{x}) = \underline{x}^{\underline{i}}$ we observe that clearly $F(\underline{x}) \geq 0$ on Δ_{n-1} and distinguish two cases:

Case: For all $\nu = 1, \ldots, n$, $i_\nu \geq 1$. Then it is clear that $F|\mathrm{bd}(\Delta_{n-1}) \equiv 0$, and hence the maximum of $F|\Delta_{n-1}$ will be assumed (only) in $\mathrm{int}\,(\Delta_{n-1})$. Using this, the multiplier method yields the claimed result.

Case: There is a ν such that $i_\nu = 0$. Then some of the x_i, say $x_1, ..., x_k$ do not occur in F. We can assume $F(\underline{x}) = x_1^{i_1} x_2^{i_2} \ldots x_k^{i_k}$ with $i_1, \ldots, i_k \geq 1$. In this case the maximum of the function F is precisely the maximum of the F restricted to the face $x_{k+1} = \ldots = x_n = 0$ of the simplex Δ_{n-1}. This is again a simplex and the multiplier method again applies as in the previous case. $\square$

We can now prove the second main result.

Theorem 3.2. Let $f(\underline{x}) = \sum f_{\underline{i}} \underline{x}^{\underline{i}} \in \mathbb{R}[\underline{x}; \deg = 2d]$. If

$$\min_{i=1,\ldots,n} f_{2d\underline{e}_i} - \frac{1}{n}(\frac{n}{2d})^{2d} \sum_{\underline{i} \in \Gamma^c} |f_{\underline{i}}| \underline{i}^{\underline{i}} + \frac{1}{n}(\frac{n}{2d})^{2d} \sum_{\underline{i} \in \Gamma} \min\{0, f_{\underline{i}}\}\, \underline{i}^{\underline{i}} \geq 0,$$

then f is sos.

Proof. For the hat of f we have by the first part of the lemma 3.1 the estimate

$$\begin{aligned}
\min_{\underline{x}\in\Delta_{n-1}} \hat{f}(\underline{x}) &= \min_{\underline{x}\in\Delta_{n-1}} (\sum_{i=1}^{n} f_{2d\underline{e}_i} x_i^{2d} - \sum_{\underline{i}\in\Gamma^c} |f_{\underline{i}}| \underline{x}^{\underline{i}} + \sum_{\underline{i}\in\Gamma} \min\{0, f_{\underline{i}}\} \underline{x}^{\underline{i}}) \\
&\geq \min_{\underline{x}\in\Delta_{n-1}} (\sum_{i=1}^{n} f_{2d\underline{e}_i} x_i^{2d}) - \sum_{\underline{i}\in\Gamma^c} |f_{\underline{i}}| \max_{\underline{x}\in\Delta_{n-1}} (\underline{x}^{\underline{i}}) + \sum_{\underline{i}\in\Gamma} \min\{0, f_{\underline{i}}\} \max_{\underline{x}\in\Delta_{n-1}} (x^{\underline{i}}). \\
&= \left(\sum_{i=1}^{n} (f_{2d\underline{e}_i})^{\frac{1}{1-2d}}\right)^{1-2d} - \sum_{\underline{i}\in\Gamma^c} \cdots + \sum_{\underline{i}\in\Gamma} \cdots .
\end{aligned}$$

Now the first summand is easily seen to be increasing in each of the $f_{2d\underline{e}_i}$. Therefore it is$\geq n^{1-2d} \min_{i=1,\ldots,n} f_{2d\underline{e}_i}$. Using this and the second part of lemma 3.1, the theorem follows via theorem 2.1. $\square$

We close with examples showing that the two piecewise linear criteria given by Lasserre and the theorem above complement each other. None is universally stronger than the other.

By theorem 2.3 in [FK] the form $\frac{1}{2}x^6 + y^6 + z^6 - f \cdot xy^2z^3$ is sos as long as $|f| \leq \sqrt{6}$.

Indeed with $f = 8/9$, the polynomial must pass the test of theorem 3.2, but its dehomogenization w.r.t. x, $\frac{1}{2} + y^6 + z^6 - \frac{8}{9}y^2z^3$ does not satisfy Lasserre's test. Conversely, taking $f = 1$, but dehomogenizing w.r.t. z, we get the polynomial $\frac{1}{2}x^6 + y^6 + 1 - xy^2$. This polynomial satisfies Lassere's test, while its homogeneous counterpart does not satisfy the criterion of theorem 3.2.

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