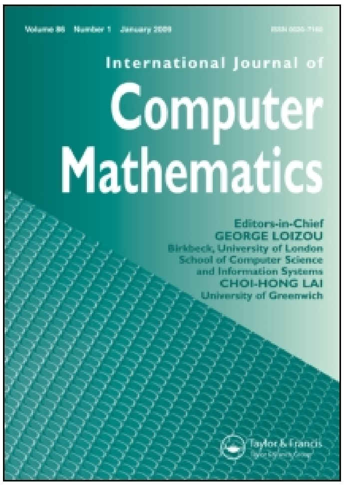

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Numerical solution for the wave equation

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In this paper we study numerical solutions for a hyperbolic system of equations using finite differences. In this setting, we propose the method of lines, with high precision in space. A class of some explicit, implicit and also semi-implicit schemes, with code variable methods, are presented. Finally, the analysis of some qualitative and quantitative proprieties of these methods is included.

Keywords: hyperbolic systems; finite differences; stability

2000 AMS Subject Classification: 65N40; 65N38

1. Introduction

Let us consider the equations of motion and continuity

$$\begin{aligned} \frac{\partial u}{\partial t} &= -g\frac{\partial \zeta}{\partial x}, \quad g \in IR \\ \frac{\partial \zeta}{\partial t} &= -\frac{\partial (Hu)}{\partial x}, \quad H(x) > 0 \end{aligned} \tag{1}$$

These equations arise in many domains of science. They are used, in particular, in the modelling of the behaviour of waves in a channel or the propagation of a tsunami. In these cases, u is the velocity, ζ denotes the surface elevation, t is the time, x denotes horizontal coordinate, g is the Earth's gravity acceleration ($g = 981\,\mathrm{cm\,s^{-2}}$), and H is depth (H can be seen as a constant or a function depending of x).

According to Kowalik's [2] analysis of this problem, we are able to control the resolution of time in the process of finding a numerical solution to this problem, but the spatial resolution depends on the available bathymetric data. To improve the quality of solutions, we shall apply higher order of approximations for the first derivatives in space. Note that this strategy is also useful for hyperbolic systems defined on a very large spatial domain.

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Higher order methods for spatial derivatives have been the object of numerous studies. Several numerical methods have been tried in the search. These include Runge–Kutta methods, optimization strategy, pseudospectral operators, finite-volume scheme – we cite some works, without being exhaustive, Zingg [9], Tam and Webb [7], Mead and Renaut [4], Schwartzkoptt *et al.* [5].

Also higher-order finite difference discretisation has to be used in both space and, Tam and Webb [7], and Djambazov *et al.* [1].

Here, we will build a class of variable formula methods for the integration of (1), which depends on two parameters. The scope of this paper is the following. In Section 2, we construct approximations, for the spatial derivatives, with high order. We establish the numerical solution for the system, in Section 3. Stability, precision and convergence are studied in Section 4. Finally, we include, in Section 5 some numerical examples.

2. Numerical approximations for the spatial derivatives

We now briefly discuss the spatial discretization of high-order. This will be useful in what follows, to approximate the derivatives that arise in the PDE.

Let us consider $y = f(x)$, where throughout this section f is a generic sufficiently differentiable function in $D \subset IR$, as well as the following grid in the domain D:

$$\cdots < x_{i-j/2} < x_i < x_{i+j/2} < \cdots$$

Here, $x_{i\pm j/2} = x_i \pm j(h/2)$ and $h = x_{i+1} - x_i$.

In order to obtain an approach of high-order for $y'(x_k)$, we can apply the following proposition:

PROPOSITION 2.1 *Let $y = f(x)$ be a sufficiently differentiable function in $D \subset IR$. Consider a partition with a mesh size $h/2$ in D.*

Then

$$y'(x_k) = f'(x_k) \approx \frac{1}{h} \sum_{\substack{j=1 \\ j \text{ odd}}}^{q} \beta_j \left[y_{k+j/2} - y_{k-j/2} \right], \tag{2}$$

is an approximation for the derivative of $y = f(x)$ in x_k, with precision $O(h^{q+1})$, where the unknowns β_j which appear in (2) are the solutions of the linear system

$$C_1 = \sum_{\substack{j=1 \\ j \text{ odd}}}^{q} j\,\beta_j = 1, \quad C_p = \sum_{\substack{j=1 \\ j \text{ odd}}}^{q} j^p\,\beta_j = 0, \quad p = 3, 5, \ldots, q,$$

where $y_k \cong y(x_k)$, $y_{k\pm j/2} \cong y(x_k \pm jh/2)$

Proof It suffices to apply the Taylor theorem. ■

2.1 *Particular cases*

In Table 1 we include some coefficients β_j, as given in (2), for methods with order $r = 2, 4, 6, 8$.

We note that for $q = 1, 3$ the approximations we obtained, as displayed above, coincide with the central difference and that referred by Kowalik in ref. [2].

Table 1. Coefficients of β_j.

q	β_1	β_3	β_5	β_7
1	1	–	–	–
3	9/8	−1/24	–	–
5	75/64	−25/384	3/640	–
7	1225/1024	−245/3072	49/5120	−5/7168

3. The method

Let us consider the hyperbolic system (1). For the sake of simplify we write it in the form

$$V_t(x,t) = AV_x(x,t) + BV(x,t), \tag{3}$$

where $V(x,t) = [u(x,t), \zeta(x,t)]^T$, $A = \begin{bmatrix} 0 & -g \\ -H & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ -\bar{H} & 0 \end{bmatrix}$, $\partial V/\partial t = V_t$, $\partial V/\partial x = V_x$ and $\bar{H} = \partial H/\partial x$.

Let the initial and boundary conditions associated with (3) be given by

$$\begin{aligned} V(x,0) &= [f_1(x,0), f_2(x,0)]^T, \\ V(0,t) &= [g_1(x,0), g_2(x,0)]^T. \end{aligned} \tag{4}$$

Assume that the domain is covered by a grid with mesh size Δx (in space) and Δt (in time). Using the approximation (2) for space and method-θ for time – see Lambert [3] – the numerical solution for (3) is given by

$$V_k^{n+1} = V_k^n + \rho\, A \left[\theta\, \delta\, V_{k,q}^n + (1-\theta)\, \delta\, V_{k,q}^{n+1}\right] + \Delta t\, B\, V_k^n, \;\; \rho = \frac{\Delta t}{\Delta x}, \tag{5}$$

where $\theta \in [0, 1]$ is a parameter. The initial points are denoted by x_0 and t_0, respectively, for space and time and

$$\delta V_{k,q}^n = \sum_{\substack{j=1 \\ j \text{ odd}}}^{q} \beta_j \left[V_{k+j/2}^n - V_{k-j/2}^n\right]. \tag{6}$$

Now, by splitting the matrix A in the form $A = A_1 + A_2$, $A_1 = \begin{bmatrix} 0 & -g \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ -H & 0 \end{bmatrix}$, the system (3) takes the form

$$\frac{\partial V}{\partial t} = A_1 \frac{\partial V}{\partial x} + A_2 \frac{\partial V}{\partial x} + BV. \tag{7}$$

The semi-implicit scheme

$$V_k^{n+1} = V_k^n + \rho\left[A_1 \delta V_{k,q}^n + A_2 \delta V_{k,q}^{n+1}\right] + \Delta t B V_k^n, \tag{8}$$

can be obtained for the solution of (3) or (7) with the initial conditions (4).

In the following sections we consider, respectively, the classes of methods defined by (5) and by (8).

4. Qualitative and quantitative behaviour: stability, precision and convergence

4.1 *Stability*

In order to study the stability of the scheme, it is convenient to observe that the matrix A is diagonalizable. In fact, there exists a matrix S such that $D = SAS^{-1}$ is a diagonal matrix,

$$S = \begin{bmatrix} 1 & 1 \\ -\frac{\sqrt{gH}}{g} & \frac{\sqrt{gH}}{g} \end{bmatrix} \quad \text{and} \quad D = \mathrm{diag}\left[\sqrt{gH} \quad -\sqrt{gH}\right].$$

We investigate the stability of the later difference schemes – (5) and (8) – by taking the discrete Fourier transform, Thomas [8].

In order to simplify the presentation, we will consider only $q = 3$ in (5) and (8). So these schemes may be written, respectively, in the form

$$V_k^{n+1} = V_k^n + \rho A\left[\theta\delta V_{k,3}^n + (1-\theta)\,\delta V_{k,3}^{n+1}\right] + \Delta t\, BV_k^n, \tag{9}$$

$$V_k^{n+1} = V_k^n + \rho\left[A_1\delta V_{k,3}^n + A_2\delta V_{k,3}^{n+1}\right] + \Delta t\, BV_k^n \tag{10}$$

The usage of the discrete Fourier transform makes it an easy task to conclude that the amplification error for (9) is given by

$$G_1(\eta) = [I - 2i\rho c(\eta)\,(1-\theta)\,A]^{-1} \times [I + 2i\rho c(\eta)\theta A + \Delta t B], \tag{11}$$

where $c(\eta) = \beta_1 \sin\eta + \beta_3 \sin 3\eta$, $\eta \in [0, 2\pi]$.

We observe that

(i) Let us $\bar{H} = 0$. Attending to the matrix A is diagonalizable, we can establish

$$G(\eta) = S^{-1}G_1(\eta)S = [I - 2i\rho c(\eta)(1-\theta)D]^{-1} \times [I + 2i\rho c(\eta)\theta D],$$

and so the eigenvalues of G are

$$\lambda_j(\eta) = \frac{1 + 2i\rho c(\eta)\theta\sqrt{gH}}{1 - 2i\rho c(\eta)(1-\theta)\sqrt{gH}}, \quad j = 1, 2. \tag{12}$$

(ii) The function $c(\eta)$ is bounded, for $\eta \in [0, 2\pi]$.

The equality (11) allows us to establish the stability of (9). We begin with the simpler case.

Proposition 4.1 *Consider the class* (9) *in the integration of problem* (1)*, with* $\bar{H} = 0$ *and together with the initial conditions given by* (4)*. If* $\theta = 1/2, 1$ *the differences scheme* (9) *is not stable*; *whilst for* $\theta = 0$ *the method is stable.*

Proof According to (12), we can verify that the eigenvalues λ_j, $j = 1, 2$, of the matrix G satisfy

(i) $|\lambda_j| > 1, \theta = 1/2, 1$
(ii) $|\lambda_j| < 1, \theta = 0$.

The proposition follows immediately. ■

We observe that if $\bar{H} \neq 0$ the Proposition above is also true. In fact, as we know – Thomas [8] – any result obtained without the B term will also be true with that term.

Furthermore, for other values of the parameter θ, $\theta \neq 0, 1/2, 1$, in order to ensure stability we must verify $|\lambda_j| < 1$, $j = 1, 2$.

Let us consider the method (10), where we can suppose, without lost of generality, that $\bar{H} = 0$. Using the discrete Fourier transform, the amplification error is

$$G_2(\eta) = [I - 2i\rho c(\eta) A_2]^{-1} \times [I + 2i\rho c(\eta) A_1]. \tag{13}$$

Regarding the stability of the scheme (10) we can state:

PROPOSITION 4.2 *Let us consider the method* (10) *in the integration of problem* (1)–(4). *If* $\rho < 6/7\sqrt{g\,H}$, *the method is stable.*

Proof The eigenvalues of the matrix G_2 are the roots λ_j, $j = 1, 2$, of the polynomial

$$\lambda^2 + \lambda \left\lfloor g\,H\,d^2(\eta) - 2 \right\rfloor + 1,$$

where $d(\eta) = \beta_1 \sin \eta + \beta_3 \sin 3\eta$, $\eta \in [0, 2\pi]$.

So, $|\lambda_j| < 1$, $j = 1, 2$ if and only if $\rho^2 < (1/d^2(\eta)\,g\,H)$. The result arises attending to $d(\eta) < \beta_1 - \beta_3 = (7/6)$, $\forall \eta \in [0, 2\pi]$. ■

Remark The analysis above can be applied in an analogous way for $q > 3$.

In particular, Proposition 4.1 is also true and in what concerns Proposition 4.2, the restriction imposed is now

$$\rho^2 < \frac{1}{gH \sum_{j=0}^{q} \left[\beta_{4j+1} - \beta_{4j+3}\right]}. \tag{14}$$

If we define the function $c(\eta, q) = \sum_{j=0}^{q} [\beta_{4j+1} \sin[(4j + 1)\eta] + \beta_{4j+3} \sin[(4j + 3)\eta]]$, $\eta \in [0, \pi]$, we can conclude that c is a non-decreasing function in the variable q. So the higher the value of q, the higher the restriction that we must impose to ρ. In other words, when q is big, Δx may also be big. This result is satisfactory, namely when we take into account that one wishes to integrate hyperbolic systems with a large spatial domain. Figure 1 shows the function c for different values of q.

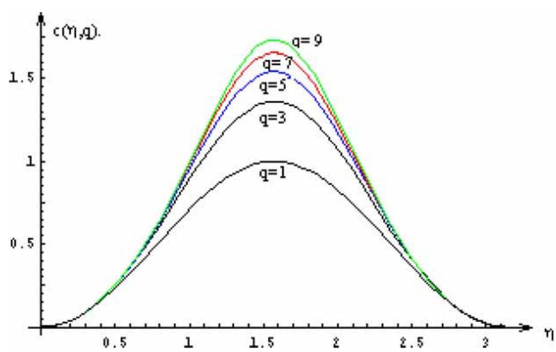

Figure 1. Function $c(\eta, q)$.

4.2 *Precision and Convergence*

We will now study the precision of the difference schemes (5) and (8).

Using a Taylor expansion in (5), we can write

$$V_k^{n+1} - V_k^n - \rho A\left\lfloor \theta\delta V_{k,q}^n + (1-\theta)\delta V_{k,q}^{n+1}\right\rfloor = \Delta t^2\left[\frac{1}{2}\frac{\partial^2 V}{\partial^2 t} - A(1-\theta)\frac{\partial^2 V}{\partial x\,\partial t}\right] + O(\Delta t^3) + O(\Delta t)O(\Delta x^{q+1})$$

where the second derivatives arising in the right bracket must be calculated in (x_k, t_n).

The difference scheme (5) then has $q+1$ order accuracy in Δx. It is second-order accurate in Δt if $\theta \neq 1/2$ and third-order accurate in Δt if $\theta = 1/2$.

Hence the scheme (5) is consistent for any value of θ and hence convergent if $\theta = 0$.

Let us consider the method (8). Using again a Taylor expansion we conclude that

$$V_k^{n+1} - V_k^n - \rho\left[A_1\delta V_{k,q}^n + A_2\delta V_{k,q}^{n+1}\right] = O(\Delta t^2) + O(\Delta t)O(\Delta x^{q+1})$$

and the precision for the scheme is $q+1$th order accurate in Δx and second order accurate in Δt respectively. So, if the value ρ satisfies (14), the scheme (8) is convergent if

$$\left|\frac{1+2i\rho c(\eta)\theta\sqrt{gH}}{1-2i\rho c(\eta)(1-\theta)\sqrt{gH}}\right| < 1,\ j = 1, 2.$$

5. Numerical results

In this section we apply the class of methods we have introduced for two problems, one of which defined on a very large domain space. A well weighted choice for the parameters θ and q, of which these methods depend, will certainly influence their computational cost. These parameters are related respectively to the stability – see Section 4.1 – and the spatial precision. Naturally, the gradient of the solution should reflect on our choice of q.

Using the classes in a practical code 'constant formula methods', that is the same method along the whole integration domain, the results may not be satisfactory/efficient, namely if one has to integrate over a large domain space. Alternatively, a pratical code 'variable formula methods' should be applied.

One may define an algorithm which arises from a code variable formula method in the following manner:

Assume that we start with a method associated to a value of ρ and q. Let δ be the prescribed tolerance.

(i) Apply the method and calculate $V_{k+1}^n - V_k^n$, $k = 1, 2, \ldots$.
If

$$\max_k \left|V_{k+1}^n - V_k^n\right| \leq \delta, \tag{15}$$

the numerical solution obtained is acceptable and the next level is considered.

(ii) If (15) does not hold in some sub domains, apply refinements at these domains (by increasing the value of (q) and repeat (i)).

5.1 *First problem*

Let us consider the problem

$$\left.\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial \zeta}{\partial x}\\ \frac{\partial \zeta}{\partial t} &= \frac{\partial u}{\partial x}\end{aligned}\right. \quad \text{in } D = \{(x,t) : x \in [0,5], t > 0\} \tag{16}$$

with the initial conditions

$$\zeta(x,0) = \begin{cases} 0, & x \leq 1 \\ 1, & 1 \leq x \leq 2 \\ 0, & \text{other cases} \end{cases} \quad \text{and} \quad u(x,0) = 1$$

We integrate the problem using the method (5), with $\Delta x = 1/10$, $\Delta t = 1/100$ and applying 0 algorithm anterior. For small gradients, we take $q = 1$ and $\theta = 1$ (explicit method), otherwise we take $q \geq 3$ and the $\theta = 0$ (implicit method).

The results obtained are included in Figure 2, where we represent in (a) the numerical solution and in (b) the numerical solution and the exact solution, for $t = 1/4$.

As can be seen, the numerical results are in agreement with what was expected. Now, for $t = 1/2$ we represent similar results in Figures 3.

We observe that we must apply the method with a refined grid in the region where the solution has a large gradient, as before.

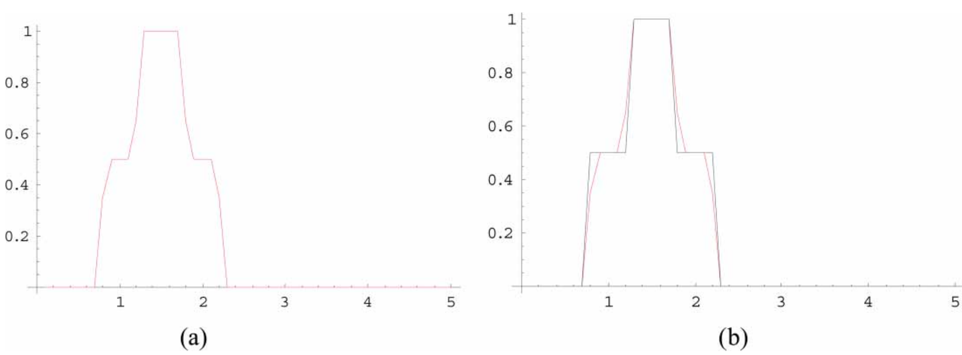

Figure 2. Exact and numerical solution, $t = 1/4$.

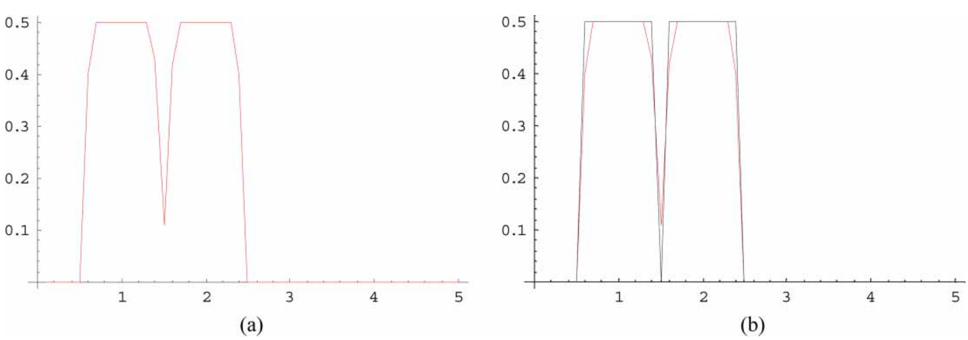

Figure 3. Exact and numerical solution, $t = 1/2$.

5.2 *Second problem*

Let us now consider the problem

$$\frac{\partial^2 u}{\partial^2 t} = \frac{\partial^2 u}{\partial^2 x} \quad \text{in } D = \{(x,t) : x \in [0, L], t > 0\} \tag{17}$$

with the initial conditions

$$u(x, 0) = e^{-20x^2}.$$

Naturally, one may rewrite Equation (17) as a system of the form (1), using an appropriate change variable.

The exact solution is represented in Figure 4 for different time levels. We took $L = 10$.

In order to solve (17) $-\ L = 1, L = 5-$ we apply the class (5) in a code constant formula methods, with $\Delta x = 1/80$ and $\Delta t = 1/100$. The results obtained are plotted in Figure 5. For a small spatial domain, the results were satisfactory. The same can not be said for a larger domain. In this case, the solution is rather imprecise, especially in the initial part of the domain.

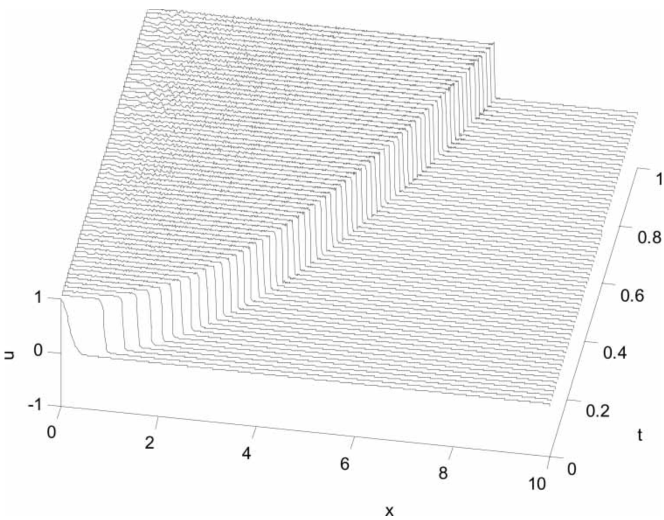

Figure 4. Problem (17): exact solution.

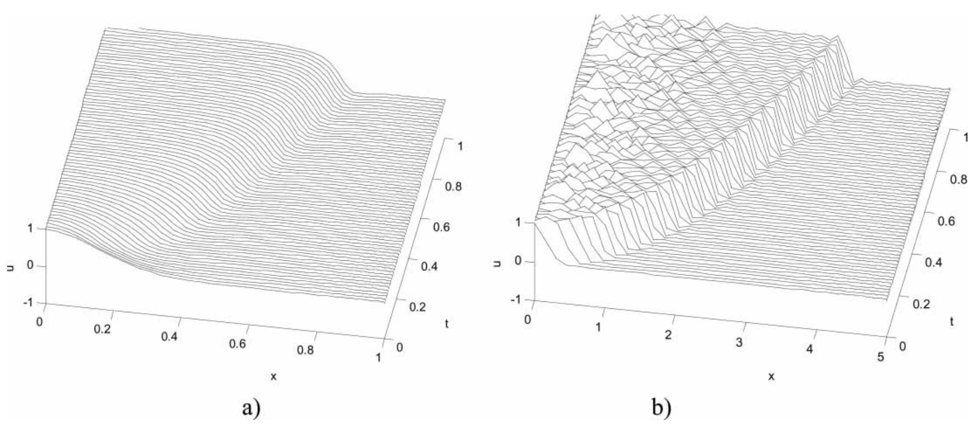

Figure 5. Problem (17): numerical solution for (17), $t = 1$.

Table 2. Error of numerical solution of (17).

L	Error (constant formula)	Error (variable formula)
1	0.050	0.0004
5	0.059	0.0004
10	0.063	0.0002

Next, we integrate (17) using code variable formula methods (the proposed algorithm). We take $\delta = 0.0001$, $\theta = 0, 1$.

Table 2 displays the error of the obtained approximation, the maximum absolute value of the difference between the exact and approximated values at $t = 2$, for different values of L. We also include the error obtained when code constant formula methods were used with $\Delta x = 1/800$ and $\Delta t = 1/100$.

6. Conclusion

The class we have presented is quite efficient when the problem one wishes to integrate has a large spatial integration domain, and when code variable formula methods are used.

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