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Long-time asymptotics for coagulation equations with injection that do not have stationary solutions

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Abstract

In this paper we study a class of coagulation equations including a source term that injects in the system clusters of size of order one. The coagulation kernel is homogeneous, of homogeneity $\gamma < 1$, such that $K(x, y)$ is approximately $x^{\gamma+\lambda}y^{-\lambda}$, when x is larger than y. We restrict the analysis to the case $\gamma + 2\lambda \geq 1$. In this range of exponents, the transport of mass toward infinity is driven by collisions between particles of different sizes. This is in contrast with the case considered in Ferreira et al. (Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire, 2023), where $\gamma + 2\lambda < 1$. In that case, the transport of mass toward infinity is due to the collision between particles of comparable sizes. In the case $\gamma + 2\lambda \geq 1$, the interaction between particles of different sizes leads to an additional transport term in the coagulation equation that approximates the solution of the original coagulation equation with injection for large times. We prove the existence of a class of self-similar solutions for suitable choices of γ and λ for this class of coagulation equations with transport. We prove that for the complementary case such self-similar solutions do not exist.

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1. Introduction

1.1. Aim of the paper

In this paper we study the long-time behavior of the coagulation equation with injection:

$$\partial_t f(t,x) = \mathbb{K}[f](t,x) + \eta(x) \tag{1.1}$$

where

$$\mathbb{K}[f](t,x) := \frac{1}{2}\int_0^x K(x-y,y)\, f(t,x-y)\, f(t,y)\,\mathrm{d}y - \int_0^\infty K(x,y)\, f(t,x)\, f(t,y)\,\mathrm{d}y, \tag{1.2}$$

and where $\eta \geq 0$. We will assume in all of what follows that $\eta(x) \not\equiv 0$, and that it is either compactly supported or it decays fast enough with the cluster size.

The study of problems with this form arises naturally in problems of aerosols and atmospheric science ([12,18,21]). In this context, the function $f(t,x)$ denotes the density of clusters with size x at time t.

The collision operator $\mathbb{K}[f](t,x)$ in (1.2) was introduced by Smoluchowski (see [19]). The kernel $K(x,y)$ encodes information about the mechanism driving the coagulation of clusters. In this paper we are interested in kernels arising in atmospheric science applications. A common feature of these kernels is the homogeneity property. Indeed, in many cases the rate of aggregation scales like a power law with the cluster size. This means that the coagulation kernel $K(x,y)$ satisfies

$$K(ax,ay) = a^\gamma K(x,y) \text{ for } a > 0 \text{ and } (x,y) \in (0,\infty)^2 \tag{1.3}$$

for some $\gamma \in \mathbb{R}$.

On the other hand, since the coagulation process does not depend on the order in which the clusters of size x and y are chosen, we have the symmetry property

$$K(x,y) = K(y,x) \text{ for } (x,y) \in (0,\infty)^2. \tag{1.4}$$

We will assume that the coagulation kernel K satisfies

$$c_2\left[\frac{x^{\gamma+\lambda}}{y^\lambda} + \frac{y^{\gamma+\lambda}}{x^\lambda}\right] \leq K(x,y) \leq c_1\left[\frac{x^{\gamma+\lambda}}{y^\lambda} + \frac{y^{\gamma+\lambda}}{x^\lambda}\right] \text{ and } \gamma + 2\lambda \geq 0, \tag{1.5}$$

with $0 < c_2 \leq c_1 < \infty$, where γ is the homogeneity parameter introduced in (1.3) and $\lambda \in \mathbb{R}$. The polynomial bounds (1.5) are satisfied by many of the most relevant collision kernels arising in aerosol and atmospheric science, such as the *diffusive* coagulation kernel and the *free molecular* kernel (see for instance [12]).

Notice that the condition $\gamma + 2\lambda \geq 0$ in (1.5) does not imply any loss of generality. This can be seen from the fact that the function $x^\alpha y^\beta + x^\beta y^\alpha$ can be written as $\left[\frac{x^{\gamma+\lambda}}{y^\lambda} + \frac{y^{\gamma+\lambda}}{x^\lambda}\right]$ with $\gamma + \lambda = \max\{\alpha, \beta\}$ and $-\lambda = \min\{\alpha, \beta\}$.

Since we will consider solutions of (1.1), (1.2) in which $f(t, \cdot)$ is a Radon measure, it is convenient to impose the following condition on the kernel K:

$$K \in \mathrm{C}((0, \infty)^2). \tag{1.6}$$

It is well known that part or all the mass of the solutions of (1.1), (1.2) can escape towards $x = \infty$ in finite, or even zero time. This phenomenon is known as *gelation*. For a more detailed analysis on this matter, see for example [3,4]. In order to guarantee that gelation does not take place, we will assume in all of what follows that

$$\gamma < 1 \text{ and } \gamma + \lambda < 1. \tag{1.7}$$

See [5] for a more detailed discussion on the gelation regimes.

We expect that gelation does not take place under the weaker assumptions $\gamma \leq 1, \gamma + \lambda \leq 1$ (see [3]). However, in the critical cases $\gamma = 1$ or $\gamma + \lambda = 1$, the self-similar solutions are not defined using power laws to scale the particle sizes but most likely using exponential functions. Given that the analysis of these solutions would require arguments different from the ones in this paper, we will not consider this case here.

The existence of solutions for Eq. (1.1), (1.2) has been considered in [6,7]. In this paper we study the long time behavior of the solutions of Eq. (1.1). Due to the presence of the source we can expect the solutions to (1.1), (1.2) to converge to a stationary non-equilibrium solution $\overline{f} = \overline{f}(x)$ as $t \to \infty$, i.e. to a solution of

$$\mathbb{K}[f](x) + \eta(x) = 0. \tag{1.8}$$

However, it turns out that if $\eta \not\equiv 0$ and $\gamma + 2\lambda \geq 1$, a solution for (1.8) does not exist. In [13], the discrete stationary coagulation model

$$\sum_{k=1}^{n-1} K_{k,n-k} f_k f_{n-k} - \sum_{k=1}^{\infty} K_{k,n} f_k f_n + \overline{\delta}_{k,n} = 0, \;\; k \geq 1 \tag{1.9}$$

has been studied for the explicit coagulation kernel

$$K_{k,n} = k^{\gamma+\lambda} n^{-\lambda} + k^{-\lambda} n^{\gamma+\lambda}, \;\; k, n \in \mathbb{N}, \tag{1.10}$$

when $|\gamma + \lambda| < 1, |\lambda| < 1, |\gamma| < 1$. Formal asymptotics for the large size behavior of the solutions f_k of (1.9) has been obtained in [13]. The results in that paper indicate that a solution of (1.9) exists if and only if $\gamma + 2\lambda < 1$.

In the case of general kernels satisfying the assumptions (1.4), (1.5), (1.6), and source terms η decreasing fast enough, it has been proved in [10] that the solutions of (1.8) (as well as its discrete counterpart) exist, if and only if $\gamma + 2\lambda < 1$.

It is worth to remark that it has been proved in [10] that the solutions of (1.8) can be estimated, up to a multiplicative constant, from above and below by the power law $x^{-\frac{3+\gamma}{2}}$ for large values of x.

Since in the case $\gamma + 2\lambda \geq 1$ a stationary solution of (1.8) does not exist, we cannot expect the solutions to (1.1), (1.2) to behave as the stationary solution $\overline{f}$ of (1.8) as $t \to \infty$ for x of order one. It is then natural to ask what is the long time asymptotics of the solutions to (1.1), (1.2) for large values of t and x. The scaling hypothesis that has been extensively used in the study of coagulation equations suggests that the mass of the particle distributions $f(t, x)$ is concentrated in cluster sizes x of order t^p for a suitable exponent p that would be determined from dimensional considerations, which take into account the way in which the mass rescales in time. In the case of kernels $K(x, y) = x^{\gamma+\lambda} y^{-\lambda} + y^{-\lambda} x^{\gamma+\lambda}$, $-1 < \lambda < 0$ with $0 \leq \gamma + \lambda < 1$ and $\gamma < 1$, it was suggested in [5], using a combination of matched asymptotics and numerical simulations, that the long time behavior of the solutions of (1.1), (1.2) is given by self-similar solutions with the form

$$f_s(t, x) = \frac{1}{t^{\frac{3+\gamma}{1-\gamma}}} \Phi(\xi), \quad \xi = \frac{x}{t^{\frac{2}{1-\gamma}}}. \tag{1.11}$$

The approximation (1.11) can be expected to be valid for large cluster sizes, i.e. $x \gg 1$. We will use from now the notation with $x \gg 1$ to indicate large cluster sizes x. In the case considered in [5] we have that $\gamma + 2\lambda < 1$ and therefore stationary solutions f_s solving (1.8) exist. In this case, (1.8) and (1.11) suggest that $\Phi(\xi)$ behaves for small values of ξ as $K\xi^{-\frac{3+\gamma}{2}}$, for a suitable constant $K > 0$. More precisely, plugging (1.11) in (1.1), (1.2), it follows that Φ solves

$$-\frac{2}{1-\gamma} \xi \Phi_{\xi} - \frac{3+\gamma}{1-\gamma} \Phi = \mathbb{K}[\Phi], \tag{1.12}$$

where Φ satisfies the following boundary condition at $\xi \to 0$ which guarantees that there is a constant flux of particles from the origin:

$$\lim_{R\to 0} \int_0^R \mathrm{d}\xi \int_{R-\xi} \mathrm{d}\eta K(\xi, \eta) \xi \Phi(\xi) \Phi(\eta) = J, \tag{1.13}$$

with $J = \int_0^\infty x\eta(x)\,\mathrm{d}x$. The existence of solutions of Eq. (1.12) satisfying the constant flux solution condition at $\xi = 0$, (1.13), has been rigorously proved in [9] for kernels K satisfying (1.4), (1.5), (1.6), (1.7) with $\gamma + 2\lambda < 1$.

The picture described above, which combines the stationary behavior $\overline{f}$, (cf. (1.8)) for cluster sizes x of order one, and the self-similar behavior (1.11) for large cluster sizes, provides a rather complete description of the long time behavior of the solutions to (1.1), (1.2) in the case $\gamma + 2\lambda < 1$. However, the same scenario cannot yield a description of the long time asymptotics of the solutions to (1.1), (1.2) if $\gamma + 2\lambda \geq 1$, because, as explained above, in this case a solution of (1.8) does not exist.

Notice that the existence/non-existence of solutions to (1.8) is related to the existence of stationary solutions of (1.8) yielding a constant flux of particles with the form of a power law, i.e. $\overline{f}(x) = cx^{-\frac{\gamma+3}{2}}$. These solutions exist for $\gamma + 2\lambda < 1$ and they do not exist for $\gamma + 2\lambda \geq 1$. In the framework of the wave turbulence it would be stated that the kernels K with $\gamma + 2\lambda < 1$ satisfy the locality property, while the kernels K with $\gamma + 2\lambda \geq 1$ do not have the locality property, see [22].

In this paper we are interested in the study of the long-time behavior of the solutions to (1.1), (1.2) in the case $\gamma + 2\lambda \geq 1$ (and the non-gelling regime $\gamma < 1$ and $\gamma + \lambda < 1$) that is described using formal asymptotics arguments. The kernel we work with is homogeneous, hence it can be expressed as

$$K(x, y) = (x + y)^{\gamma} F\left(\frac{x}{x + y}\right), \quad F(s) = F(1 - s) \quad \text{for } s \in (0, 1). \tag{1.14}$$

Notice that (1.5) implies that $\overline{c}_1 \leq F(s) \leq \overline{c}_2$ for some $\overline{c}_1, \overline{c}_2 > 0$. We will assume in what follows a condition that is more restrictive than (1.5), namely

$$\lim_{s \to 0^+} \left[s^{\lambda} F(s)\right] = 1. \tag{1.15}$$

In order to prove that in the case $\gamma + 2\lambda \geq 1$ there are no solutions of (1.8), the main idea used in [10] is based on the fact that, for this range of exponents, the transfer of clusters of size x of order one towards very large cluster sizes is so fast that the concentration of clusters with size of order one would become zero.

In the case of time dependent solutions having initially finite mass, this increases linearly due to the fact that

$$\partial_t \left(\int_0^{\infty} x f(t, x) \mathrm{d}x\right) = \int_0^{\infty} x \eta(x) \mathrm{d}x.$$

For large times, due to the increase of the average cluster size, we might expect that, if $\gamma + 2\lambda \geq 1$, there should be a fast transport of the newly injected clusters of order one towards much larger cluster sizes. This almost instantaneous transport results in small concentrations of clusters of order one for large times.

We now remark that the part of the coagulation operator which describes the coagulation between particles of different sizes can be approximated by means of a transport operator in the space of cluster sizes. These arguments, that will be described in detail using formal asymptotics in Sect. 2, show that the solutions of (1.1), (1.2) for large cluster sizes can be approximated by means of the following equation if $\gamma + 2\lambda \geq 1$:

$$\partial_t f(t, x) + \frac{\partial_x \left(x^{\gamma+\lambda} f(t, x)\right)}{\int_0^{\infty} z^{\gamma+\lambda} f(t, z) \mathrm{d}z} = \mathbb{K}[f](t, x), \quad \text{for } x \gg 1. \tag{1.16}$$

We emphasize that the non-local transport term $\frac{\partial_x \left(x^{\gamma+\lambda} f(t,x)\right)}{\int_0^{\infty} z^{\gamma+\lambda} f(t,z) \mathrm{d}z}$ is a consequence of the presence of the source $\eta(x)$ in (1.1). The fact that the contributions of the coagulation operator $\mathbb{K}[f]$ which are due to the aggregation of particles with very different sizes can be approximated by a differential operator has been extensively used in the literature of coagulation equations (cf. [12]). The resulting first order terms are often referred to represent *heterogeneous condensation* (cf. [12]). On the other side, the term $\mathbb{K}[f]$ in (1.16) describes the aggregation of clusters of comparable sizes.

The solutions of (1.16) are expected to describe the asymptotic behavior of the solution of Eq. (1.1) both when $\gamma + 2\lambda > 1$ and when $\gamma + 2\lambda = 1$, even if in these two cases we have two slightly different scenarios. Namely, when $\gamma + 2\lambda > 1$, we will have that if x is of order 1, then $f(t, x) \to 0$ as $t \to \infty$, while when $\gamma + 2\lambda = 1$ we will have that if x is of order 1, then $f(t, x) \to f_s(x)$ as $t \to \infty$, where f_s is a solution of

$$\mathbb{K}[f](x) + \eta(x) - x^{-\lambda} f(x) = 0 \tag{1.17}$$

(compare with (1.8)). Due to the presence of the term $x^{-\lambda} f(x)$, Eq. (1.17) might have solutions when $\gamma + 2\lambda = 1$.

The scaling properties of (1.16) suggest that this equation is compatible with the existence of self-similar solutions for Eq. (1.16) with the form

$$f_s(t, x) = \frac{1}{t^{\frac{3+\gamma}{1-\gamma}}} \Phi\left(\frac{x}{t^{\frac{2}{1-\gamma}}}\right). \tag{1.18}$$

Here the self-similar profile Φ satisfies the following equation, obtained by substituting equality (1.18) in Eq. (1.16) and using the self-similar change of variables $\xi = \frac{x}{t^{\frac{2}{1-\gamma}}}$,

$$\begin{aligned} &-\tfrac{3+\gamma}{1-\gamma}\Phi(\xi) - \tfrac{2}{1-\gamma}\xi\partial_\xi\Phi(\xi) + \tfrac{1}{\int_0^\infty \eta^{\gamma+\lambda}\Phi(\eta)d\eta} \\ &\quad\times \tfrac{\partial}{\partial\xi}\left(\xi^{\gamma+\lambda}\Phi(\xi)\right) = \mathbb{K}[\Phi](\xi), \quad \xi > 0. \end{aligned} \tag{1.19}$$

The main result of this paper is to determine the range of exponents γ and λ satisfying $\gamma + 2\lambda \geq 1$ and the non-gelation conditions (1.7) for which self-similar solutions of (1.16) with the form (1.18) exist (see Fig. 1 for a classification of these exponents). Specifically, we will prove the following: suppose that $\gamma + 2\lambda \geq 1$ and that (1.7) holds. Then

- If $\gamma > -1$, there exists at least one self-similar solution of (1.16) with the form (1.18).
- If $\gamma \leq -1$ and $\gamma + 2\lambda > 1$, no solutions of (1.16) with the form (1.18) exist (See Fig. 1).
- If $\gamma \leq -1$ and $\gamma + 2\lambda = 1$, we prove that there are no self-similar solutions f_s of the form (1.18) such that $\int_0^1 x^{-\lambda}\Phi(x)\mathrm{d}x < \infty$.

The meaning of the condition $\int_0^1 x^{-\lambda}\Phi(x)\mathrm{d}x < \infty$ is that the number of clusters removed by the coagulation process in any bounded time interval is finite. We therefore do not exclude the existence of a self-similar solution of Eq. (1.16) with $\int_0^1 x^{-\lambda}\Phi(x)\mathrm{d}x = \infty$.

A remarkable property of the self-similar solutions of (1.16), that we constructed in this paper, is that they vanish identically for $0 < \xi < \rho(M_{\gamma+\lambda}) := \left(\frac{1-\gamma}{2M_{\gamma+\lambda}}\right)^{\frac{1}{1-\gamma-\lambda}}$, where

$$M_{\gamma+\lambda} := \int_{(0,\infty)} \xi^{\gamma+\lambda}\Phi(\xi)\mathrm{d}\xi.$$

The fact solutions Φ of (1.19) vanish in an interval $(0, \rho(M_{\gamma+\lambda}))$ means that, for large times t, the injected particles are transferred almost instantaneously to clusters with sizes $x \geq \rho(M_{\gamma+\lambda})t^{\frac{2}{1-\gamma}}$. Moreover, the fraction of clusters with sizes $x < \rho(M_{\gamma+\lambda})t^{\frac{2}{1-\gamma}}$ becomes negligible for very long times. The existence of this "minimal" cluster size for large times is a remarkable feature that, to our knowledge, has not been observed in the literature on self-similar solutions for the coagulation equation (see for instance [7,8,11,17,20]). In particular, it is worth to notice that this behavior of the self-similar solutions is very different from the one exhibited by the self-similar solutions obtained in the case $\gamma + 2\lambda < 1$ in [9].

The results that we obtain in this paper in the critical case $\gamma + 2\lambda = 1$ are more fragmentary than those obtained for $\gamma + 2\lambda > 1$. In the case $\gamma + 2\lambda = 1$, we only prove the existence of a solution Φ of (1.19) for $\gamma > -1$ with $\Phi = 0$ on $(0, \rho(M_{\gamma+\lambda}))$, but we do not prove that any solution vanishes in the interval $(0, \rho(M_{\gamma+\lambda}))$.

On the other hand, we will prove in this paper that the function Φ which describes the self-similar profile in (1.18) decreases exponentially as $\xi \to \infty$ in the same manner as the self-similar solutions constructed in [9] and as it usually happens for the self-similar solutions of coagulation equations in problems without injection, see [7,11].

As indicated above, if $\gamma \leq -1$ and $\gamma+2\lambda > 1$, there are no self-similar solutions of (1.16) with the form (1.18) and, if $\gamma \leq -1$ and $\gamma + 2\lambda = 1$, there are no self-similar solutions of (1.16) with the form (1.18) satisfying $\int_0^1 x^{-\lambda}\Phi(x)\mathrm{d}x < \infty$. It is natural to ask what is the long time asymptotics of the solutions of (1.1), (1.2) in this case. This question will be the subject of study of a future work.

In both cases, the coagulation term $\mathbb{K}[f]$ in (1.16) which describes the aggregation of particles with comparable sizes is negligible for large times, and the long time behavior of the distribution of clusters is determined by the coagulation of particles of size x of order one with large particles. The main difference between the cases $\gamma + \lambda \geq 0$ and $\gamma + \lambda < 0$ arises from the fact that in the second case the transfer of clusters (not monomers) from the region where x is of order one to $x \gg 1$, *heterogeneous condensation*, is relevant.

There are several results in the physical literature which are related, and are consistent with the ones in this paper; see for instance [2,14–16]. In [2] a coagulation model with kernels satisfying (1.5) with parameters γ, λ such that $\gamma + 2\lambda \geq 1$ and including also a source term and a removal of particles term has been studied. In that problem, the cluster concentrations as $t \to \infty$ are determined by the coalescence of particles of very different sizes. It is then possible to approximate the coagulation-

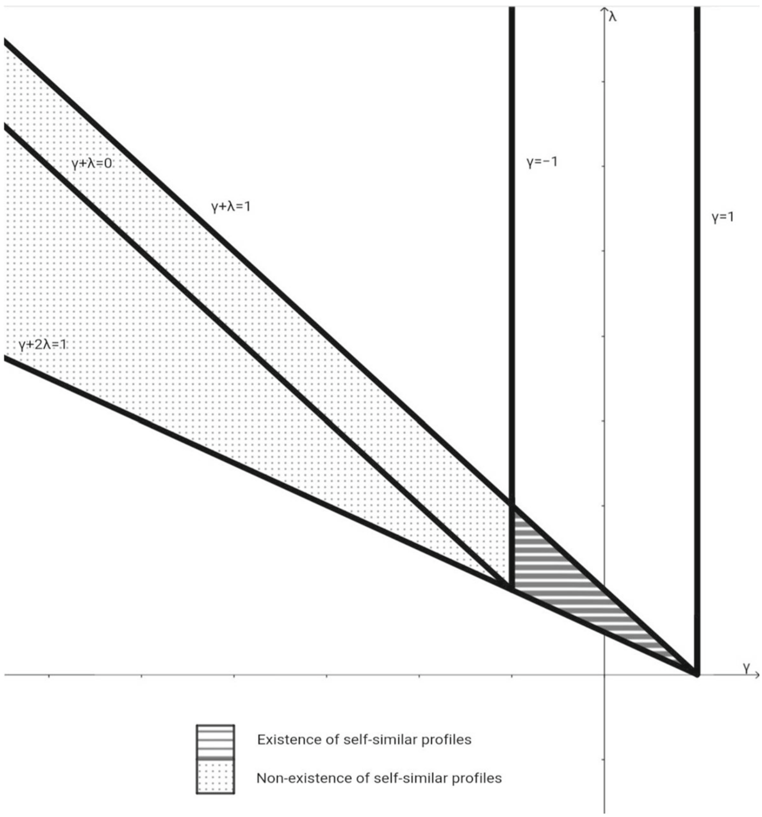

Fig. 1. Coefficients for existence and non-existence of self-similar profiles in the case $\gamma + 2\lambda > 1$

removal model for large clusters by means of the equation

$$\partial_t f(t,x) = -\partial_x \left(x^{\gamma+\lambda} f(t,x)\right) + \int_1^{x/2} y^{1-\lambda} f(t,y)\mathrm{d}y \\ - f(t,x)\int_x^K y^{\gamma+\lambda} f(t,y)\mathrm{d}y,$$

where K is the maximum particle size in the system. The numerical simulations in [2] show that the concentration of cluster sizes of order one approach to a stationary solution that converges to zero if K is sent to infinity.

In [14] a coagulation model with injection and with kernels satisfying (1.5) with $\gamma = 0$ and $\lambda \in (1/2, 1]$ has been considered. Numerical simulations and formal computations in [14] suggest that the cluster concentration for clusters of order one tends to zero as $t \to \infty$. In addition, it is suggested that the concentration for large clusters are described by a self-similar solution.

In [15,16] coagulation equations with injection are considered with kernels satisfying (1.5) with $-\lambda = \gamma \leq -1$. It is seen there that for $\gamma + 2\lambda \geq 1$, the solutions of the corresponding coagulation equation behave in a non self-similar manner and decay logarithmically. This is in agreement with the non-existence of self-similar solutions that we obtained in this paper for $\gamma \leq -1$.

1.2. Notation and plan of the paper

We use the notation $\mathbb{R}_* := (0, \infty)$ and $\mathbb{R}_+ := [0, \infty)$. Given an interval $I \subset \mathbb{R}_+$ we denote with $C_c(I)$ the Banach space of the functions on I that are continuous and compactly supported. We endow the space $C_c(\mathbb{R}_*)$ with the supremum norm denoted with $\| \cdot \|_\infty$. For a function ϕ, if $\phi(x) = 0$, for all x in an interval I, we will denote it by $\phi(I) = 0$. We keep the same notation to mean that the support of a measure is outside the interval I. We denote with $\mathcal{M}_+(I)$ the space of the non-negative Radon measures on I. Given a measure $\mu \in \mathcal{M}_+(I)$ we denote with $\|\mu\|_{TV}$ the total variation norm of μ. For a compact interval $I \subseteq \mathbb{R}_*$ and two bounded measures μ, ν, we denote the Wasserstein metric by W_1, namely

$$W_1(\mu, \nu) = \sup_{||\varphi||_{\text{Lip}} \leq 1} \int_I \varphi(x)(\mu - \nu)(\mathrm{d}x), \tag{1.20}$$

where the supremum is taken over the Lipschitz functions and where $\|f\|_{\text{Lip}} = \|f\|_\infty + [f]_{\text{Lip}}$, with $[f]_{\text{Lip}} = \sup_{\substack{x,y \in I \\ x \neq y}} \frac{|f(x)-f(y)|}{|x-y|}$.

To keep the notation lighter, we denote with C and c constants that might change from line to line in the computations. In addition, we use the notation $f \lesssim g$, for two functions f, g, to mean that there exists a constant $C > 0$ such that $f \leq Cg$.

Moreover, given a measure μ, we denote with $M_\alpha(\mu)$ the α moment of μ, i.e.

$$M_\alpha(\mu) := \int_{\mathbb{R}_*} x^\alpha \mu(\mathrm{d}x).$$

To simplify the notation, in some cases, we write M_α instead $M_\alpha(\mu)$ if the choice of the measure μ is clear in the argument.

We use the notation $f \sim g$ as $x \to x_0$ to indicate that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$, while we use the notation $f \approx g$ to say that there exists a constant $M > 0$ such that

$$\frac{1}{M} \leq \frac{f}{g} \leq M.$$

As previously mentioned, we use the notation $x \gg 1$ for large cluster sizes x. Additionally, for cluster sizes x, y, we denote $x \gg y$ or $y \ll x$ to mean that x is much larger than y. For two terms, A and B, we use the notation $A \simeq B$ to mean informally that A can be approximated in terms of B in the region under consideration in the respective formula.

The rest of this paper is organized as follows: in Sect. 2 we present a heuristic motivation to study the existence of the self-similar solutions considered in this paper. In Sect. 3 we present the main results of the paper regarding the existence and non-existence of a self-similar profile and its properties. In Sect. 4 we explain the main ideas behind the proofs of existence and non-existence, skipping the technical difficulties of the proofs. Section 5 deals with the proof of the existence of a self-similar solution for Eq. (1.19) when $\gamma > -1$. We also prove in this section that the solution decays exponentially for large values. Section 6 deals with the non-existence of self-similar solutions for Eq. (1.19) when $\gamma \leq -1$.

2. Asymptotic description of the long time behavior

2.1. The case $\gamma + 2\lambda > 1$

In this section we describe the long time asymptotics of the solutions of (1.1), (1.2) if $\gamma + 2\lambda > 1$ using formal asymptotic arguments.

As indicated in the Introduction, we expect $f(t, x)$ to converge to zero as $t \to \infty$ for x of order one. Therefore, the contribution due to the term $\frac{1}{2}\int_0^x K(x-y, y) f(t, x-y) f(t, y)\,dy$ can be expected to be negligible in this region since this term is quadratic in f and we can expect the linear term $\int_0^\infty K(x, y) f(t, y)\,dy f(t, x)$ to give a larger contribution. We will check that these assumptions are self-consistent, in the sense that they will predict an asymptotic behavior for f for which the assumptions made hold.

We examine the asymptotic behavior of the linear term $\int_0^\infty K(x, y) f(t, y)\,dy f(t, x)$ when x is of order one. Due to the effect of the coagulation, we expect the distribution $f(t, y)$ to be concentrated for long times in the larger cluster sizes y as $t \to \infty$. Using the assumptions (1.14) and (1.15), we obtain the following asymptotic behavior of $K(x, y)$ for $x \ll y$:

$$K(x, y) \simeq y^{\gamma+\lambda} x^{-\lambda}. \tag{2.1}$$

We then expect to have the following asymptotics as $t \to \infty$, due to the concentration of f in the large cluster sizes

$$\int_0^\infty K(x, y) f(t, y)\,dy \sim x^{-\lambda}\int_0^\infty y^{\gamma+\lambda} f(t, y)\,dy \quad \text{as} \quad t \to \infty$$

for x of order one. Then, considering the dominant terms in (1.1), (1.2) for x of order one, we obtain the following equation:

$$\partial_t f(t, x) = -x^{-\lambda} f(t, x)\int_0^\infty y^{\gamma+\lambda} f(t, y)\,dy + \eta(x). \tag{2.2}$$

As explained in the introduction, if $\gamma + 2\lambda > 1$, since steady states describing the cluster concentrations with x of order one do not exist, we expect to have $f(t, x) \to 0$ as $t \to \infty$. This suggest that we should have $M_{\gamma+\lambda} = \int_0^\infty y^{\gamma+\lambda} f(t, y)\,dy \to \infty$ as $t \to \infty$. Suppose that $\partial_t M_{\gamma+\lambda} \ll M_{\gamma+\lambda}$ as $t \to \infty$ (something that would happen if $M_{\gamma+\lambda}$ behaves like a power law, as we will see to be the case). Then (2.2) implies the following asymptotic behavior for $f(t, x)$:

$$f(t, x) \sim \frac{x^\lambda \eta(x)}{\int_0^\infty y^{\gamma+\lambda} f(t, y)\,dy} \quad \text{as} \quad t \to \infty \tag{2.3}$$

for x of order one. We notice that the higher order contributions to (2.3) due to the contributions of the term $\frac{1}{2}\int_0^x K(x-y,y)\, f(t,x-y)\, f(t,y)\,\mathrm{d}y$ are of order $\frac{1}{(M_{\gamma+\lambda})^2}$ or smaller. Therefore, these contributions will be neglected in what follows.

Equation (2.3) yields an approximate formula for the concentration of clusters with x of order one. We now approximate the part of the coagulation operator which is due to the aggregation of particles with size one with very large particles. To this end we introduce a characteristic length $L = L(t) \gg 1$ such that we can approximate $f(t,y)$ by means of (2.3) for $y \leq L$. We attempt to approximate the evolution of the distribution $f(t,x)$ for $x \gg L$. Then, the coagulation operator in (1.2) can be approximated, using the symmetry properties of the first term on the right of (1.2), as follows:

$$
\begin{aligned}
\mathbb{K}[f](t,x) &= \frac{1}{2}\int_0^x K(x-y,y)\, f(t,x-y)\, f(t,y)\,\mathrm{d}y \\
&\quad - \int_0^\infty K(x,y)\, f(t,x)\, f(t,y)\,\mathrm{d}y \\
&= \left[\int_0^L K(x-y,y)\, f(t,x-y)\, f(t,y)\,\mathrm{d}y \right.\\
&\quad \left. - \int_0^L K(x,y)\, f(t,x)\, f(t,y)\,\mathrm{d}y\right] \\
&\quad + \left[\frac{1}{2}\int_L^{x-L} K(x-y,y)\, f(t,x-y)\, f(t,y)\,\mathrm{d}y \right.\\
&\quad \left. - \int_L^\infty K(x,y)\, f(t,x)\, f(t,y)\,\mathrm{d}y\right].
\end{aligned}
$$

We can rewrite this formula as

$$
\begin{aligned}
\mathbb{K}[f](t,x) \ &= \left[\int_0^L \left[K(x-y,y)\, f(t,x-y) - K(x,y)\, f(t,x)\right] f(t,y)\,\mathrm{d}y\right] \\
&\quad + \mathbb{K}\left[f\chi_{[L,\infty)}\right](t,x),
\end{aligned}
\tag{2.4}
$$

where $\chi_{[L,\infty)}$ denotes the characteristic function of the interval $[L,\infty)$. We now approximate $\mathbb{K}[f](t,x)$ for large values of x and $t \to \infty$ and more precisely for $x \gg L$. To this end we use (2.3) and we assume also that K and f are sufficiently regular for large values of x. Then, using the fact that $y \ll x$ we obtain the approximation

$$
K(x-y,y)\, f(t,x-y) - K(x,y)\, f(t,x) \simeq -y\frac{\partial}{\partial x}\left[K(x,y)\, f(t,x)\right]. \tag{2.5}
$$

Moreover, (2.1) yields an approximation for $K(x,y)$ if $x \ll y$. Exchanging the roles of x and y we obtain $K(x,y) \sim x^{\gamma+\lambda}y^{-\lambda}$ as $\frac{x}{y} \to \infty$ and plugging this formula into (2.5) we obtain

$$
K(x-y,y)\, f(t,x-y) - K(x,y)\, f(t,x) \simeq -y^{1-\lambda}\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f(t,x)\right].
$$

Using this approximation in (2.4) we then obtain

$$\mathbb{K}[f](t,x) = -\left[\int_0^L y^{1-\lambda} f(t,y)\,\mathrm{d}y\right]\left[\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f(t,x)\right]\right] + \mathbb{K}\left[f\chi_{[L,\infty)}\right](t,x)$$

for $x \gg 1$ and $t \to \infty$. We can now use (2.3) to derive a formula for $\int_0^L y^{1-\lambda} f(t,y)\,\mathrm{d}y$. We then obtain the approximation

$$\mathbb{K}[f](t,x) \simeq -\frac{\int_0^\infty y\eta(y)\,\mathrm{d}y}{\int_0^\infty y^{\gamma+\lambda} f(t,y)\,\mathrm{d}y}\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f(t,x)\right] + \mathbb{K}\left[f\chi_{[L,\infty)}\right](t,x). \tag{2.6}$$

Notice that we use that $\int_0^L y\eta(y)\,\mathrm{d}y \simeq \int_0^\infty y\eta(y)\,\mathrm{d}y$ since by assumption $\eta(y)$ decreases sufficiently fast for large values of y.

We will denote as f_{out} the distribution of particles in the region where $x \gg 1$. More precisely we write $f_{out} = f\chi_{[L,\infty)}$. Combining (1.1), (1.2) with (2.6) we obtain the following evolution equation for f_{out}:

$$\partial_t f_{out}(t,x) + \frac{\int_0^\infty y\eta(y)\,\mathrm{d}y}{\int_0^\infty y^{\gamma+\lambda} f_{out}(t,y)\,\mathrm{d}y}\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f_{out}(t,x)\right] = \mathbb{K}[f_{out}](t,x). \tag{2.7}$$

Notice that we use the approximation $\int_0^\infty y^{\gamma+\lambda} f(t,y)\,\mathrm{d}y \simeq \int_0^\infty y^{\gamma+\lambda} f_{out}(t,y)\,\mathrm{d}y$ that might be expected because $f(t,y) \to 0$ for $y \leq L$ as $t \to \infty$ (cf. (2.3)).

In the rest of the paper we will study the properties of the self-similar solutions associated to Eq. (2.7). It is worth to remark that the transport term on the left of (2.7) is the way in which the injection of particles with size x of order one affects the outer distribution of clusters f_{out}. Indeed, multiplying (2.7) by x and integrating we obtain

$$\partial_t\left(\int_0^\infty x f_{out}(t,x)\,\mathrm{d}x\right) + \frac{\int_0^\infty y\eta(y)\,\mathrm{d}y}{\int_0^\infty y^{\gamma+\lambda} f_{out}(t,y)\,\mathrm{d}y}\int_0^\infty x\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f_{out}(t,x)\right]\mathrm{d}x$$
$$= \int_0^\infty x\mathbb{K}[f_{out}](t,x)\,\mathrm{d}x. \tag{2.8}$$

The mass conservation property associated to the coagulation kernel yields

$$\int_0^\infty x\mathbb{K}[f_{out}](t,x)\,\mathrm{d}x = 0.$$

On the other hand, integrating by parts in the second term on the left of (2.8) we obtain $\int_0^\infty x\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f_{out}(t,x)\right]\mathrm{d}x = -\int_0^\infty x^{\gamma+\lambda} f_{out}(t,x)\,\mathrm{d}x$. Combining these results we obtain

$$\partial_t\left(\int_0^\infty x f_{out}(t,x)\,\mathrm{d}x\right) = \int_0^\infty x\eta(x)\,\mathrm{d}x. \tag{2.9}$$

The identity (2.9) states that the total mass of the clusters in the outer region is equal to the injection rate. This result, that holds for long times, could be expected because in the regime described in this section, the injected particles are transferred instantaneously to large cluster sizes. This is also consistent with $f(t,x) \to 0$ as $t \to \infty$ when $x \approx 1$.

2.2. *The case* $\gamma + 2\lambda = 1$

In the case $\gamma + 2\lambda = 1$, the approximation of the concentrations of clusters $f(t,\cdot)$ in the region where x is of order one must be obtained in a different manner. The reason is that in this case we cannot expect the moment $M_{\gamma+\lambda} = \int_0^\infty x^{\gamma+\lambda} f(t,x)\mathrm{d}x$ to converge to infinity as $t \to \infty$. Indeed, suppose that most of the mass of the monomers is distributed in a characteristic length $L(t)$ that increases as $t \to \infty$. Then, if we denote as M_0 and M_1 the moments of f of order zero and one, respectively, (i.e. $\int_0^\infty f(t,x)\,\mathrm{d}x$ and $\int_0^\infty xf(t,x)\,\mathrm{d}x$, respectively), we have that $M_1 = M_0 L(t)$ and $M_{\gamma+\lambda} = M_0 L(t)^{\gamma+\lambda}$.

Assuming that $\int_0^\infty x\eta(x)\,\mathrm{d}x = 1$ and hence that $M_1 \simeq t$ as $t \to \infty$, we deduce that $M_0 L(t) = t$. The rescaling properties of (1.1), (1.2) suggest that $\frac{M_0}{t} \simeq (M_0)^2 L(t)^\gamma$. Therefore, plugging the identity $M_0 = \frac{t}{L(t)}$ in this formula we obtain $L(t) = t^{\frac{2}{1-\gamma}}$. Hence

$$M_{\gamma+\lambda} = M_0 L(t)^{\gamma+\lambda} = tL(t)^{\gamma+\lambda-1} = t^{1+\frac{2}{1-\gamma}(\gamma+\lambda-1)} = 1. \tag{2.10}$$

It then follows that the self-similar rescaling ansatz implies that $M_{\gamma+\lambda}$ remains of order one for large times. A consequence of this is that we cannot approximate (1.1), (1.2) for clusters of order one by means of Eq. (2.2). Instead of this we will use a different approximation by splitting f in an outer part which describes the cluster distribution for x of order L and an inner part that describes the cluster distribution for x of order one. More precisely, we write

$$f(t,x) = f_{inner}(t,x) + f_{outer}(t,x), \tag{2.11}$$

where $f_{outer} = f\chi_{[L,\infty)}$, while $f_{inner} = f\chi_{(0,L]}$, for a constant $L > 0$.

Using (1.1), (1.2) we would then obtain the following evolution equation for f_{inner}:

$$\begin{aligned}\partial_t f_{inner}(t,x) &= \mathbb{K}[f_{inner}](t,x) - f_{inner}(t,x)\int_0^\infty K(x,y)\,f_{outer}(t,y)\,\mathrm{d}y \\ &\quad+\eta(x). \end{aligned}\tag{2.12}$$

Here we have used the decomposition (2.11) in the loss term of the coagulation operator $\mathbb{K}[f]$. We use also the fact that in order to compute the gain term for x of order one we need to use only f_{inner}.

By assumption, the main contribution of $f_{outer}(t,y)$ is due to clusters with size $L \gg 1$. On the other hand, for x of order one, we have the approximation

$$K(x,y)\,f_{outer}(t,y) \simeq y^{\gamma+\lambda}x^{-\lambda}f_{outer}(t,y)$$

and (2.12) becomes

$$\partial_t f_{inner}(t,x) = \mathbb{K}[f_{inner}](t,x) - f_{inner}(t,x)\,x^{-\lambda}\int_0^\infty y^{\gamma+\lambda} f_{outer}(t,y)\,\mathrm{d}y + \eta(x)\,. \tag{2.13}$$

The scaling argument above, (2.10), suggests that $\int_0^\infty y^{\gamma+\lambda} f_{outer}(t,y)\,\mathrm{d}y$ approaches to a positive constant as $t\to\infty$ if f_{outer} behaves in a self-similar manner and $\gamma+2\lambda=1$. We will write

$$M_{\gamma+\lambda} = \int_0^\infty y^{\gamma+\lambda} f_{outer}(t,y)\,\mathrm{d}y. \tag{2.14}$$

On the other hand, we will assume that the function f_{inner}, which is described by means of (2.13), approaches to a stationary solution for large times. We then obtain the following equation which would be expected to describe the behavior of the solutions of (2.13) for long times

$$\mathbb{K}[f_{inner}](x) - M_{\gamma+\lambda} f_{inner}(x)\,x^{-\lambda} + \eta(x) = 0. \tag{2.15}$$

We can now derive an equation describing the evolution of f_{outer}. To this end we argue as in the case of $\gamma+2\lambda>1$ in order to approximate the effect in f_{outer} due to the collisions of clusters with size L with clusters with size of order one. Using approximations analogous to the ones used for the derivation of (2.7) (cf. (2.4), (2.5)) we obtain the following approximation for $\mathbb{K}[f](t,x)$:

$$\mathbb{K}[f](t,x) \simeq -\left[\int_0^\infty y^{1-\lambda} f_{inner}(t,y)\,\mathrm{d}y\right]\left[\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f_{outer}(t,x)\right]\right] + \mathbb{K}[f_{outer}](t,x) \tag{2.16}$$

for x of order L.

On the other hand, multiplying (2.15) by x and integrating in $(0,\infty)$ we obtain, since f_{inner} is zero for $x>L$, that

$$M_{\gamma+\lambda}\int_0^\infty y^{1-\lambda} f_{inner}(y)\,\mathrm{d}y = \int_0^\infty y\eta(y)\,\mathrm{d}y.$$

Using this formula to eliminate $\int_0^\infty y^{1-\lambda} f_{inner}(y)\,\mathrm{d}y$ in (2.16) we obtain

$$\mathbb{K}[f](t,x) \simeq -\frac{\int_0^\infty y\eta(y)\,\mathrm{d}y}{M_{\gamma+\lambda}}\left[\frac{\partial}{\partial x}\left[x^{\gamma+\lambda} f_{outer}(t,x)\right]\right] + \mathbb{K}[f_{outer}](t,x)\,.$$

Therefore, we obtain that f_{outer} satisfies Eq. (2.7).

The whole asymptotic behavior derived here relies on the existence of solutions of Eq. (2.15). In this equation, the value of $M_{\gamma+\lambda}$ is chosen as one associated to a self-similar solution of (2.7). Equation (2.15) can then be interpreted as a stationary solution for a coagulation equation with source η and a removal term $-M_{\gamma+\lambda} f_{inner}(x)\,x^{-\lambda}$. The results in [10] imply that no solutions of (2.15) exist if $M_{\gamma+\lambda}=0$ and $\gamma+2\lambda=1$, but when $M_{\gamma+\lambda}>0$ the existence/non-existence of stationary solutions for Eq. (2.15) is still an open problem.

3. Setting and main results

The main results of this paper concern the existence and the non-existence of solutions to the following equation:

$$-\tfrac{3+\gamma}{1-\gamma}\Phi(\xi)-\tfrac{2}{1-\gamma}\xi\Phi_{\xi}(\xi)+\tfrac{1}{\int_0^\infty \eta^{\gamma+\lambda}\Phi(\eta)d\eta}\times\tfrac{\partial}{\partial\xi}\left(\xi^{\gamma+\lambda}\Phi(\xi)\right)=\mathbb{K}[\Phi](\xi),\quad \xi>0. \tag{3.1}$$

We now write precisely what we mean by a solution of Eq. (3.1) and then we state the main theorems on the existence of a self-similar profile under certain assumptions on the parameters γ and λ (Theorem 5.1), on its properties (Theorem 6.1) and on the non-existence of the self-similar solutions under different assumptions on the parameters γ and λ (Theorem 3.5).

Definition 3.1. Let K be a homogeneous symmetric coagulation kernel satisfying (1.5), (1.6), with homogeneity $\gamma<1$ and with $\gamma+\lambda<1$ and $\gamma+2\lambda\geq 1$. A self-similar profile of Eq. (2.7) with respect to the kernel K, is a measure $\Phi\in\mathcal{M}_+(\mathbb{R}_*)$ such that

$$0<\int_{\mathbb{R}_*}x^{\gamma+\lambda}\Phi(\mathrm{d}x)<\infty \text{ and } \int_{(0,1)}x^{1-\lambda}\Phi(\mathrm{d}x)<\infty, \tag{3.2}$$

and such that it satisfies the equation

$$\int_{\mathbb{R}_*}\varphi'(x)\left[\frac{2}{1-\gamma}x-\frac{x^{\gamma+\lambda}}{\int_{\mathbb{R}_*}z^{\gamma+\lambda}\Phi(\mathrm{d}z)}\right]\Phi(\mathrm{d}x)-\frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*}\varphi(x)\Phi(\mathrm{d}x)=\frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}K(x,y)\left[\varphi(x+y)-\varphi(x)-\varphi(y)\right]\Phi(\mathrm{d}x)\Phi(\mathrm{d}y) \tag{3.3}$$

for every test function $\varphi\in C_c^1(\mathbb{R}_*)$.

Remark 3.2. For every test function $\varphi\in C_c^1(\mathbb{R}_*)$ (hence such that $\varphi=0$ near zero) all the integrals in Eq. (3.3) are finite. The integrals in the left-hand side of Eq. (3.3) are bounded due to the fact that φ is compactly supported and that Φ is a Radon measure. We analyse now the right hand side. Since $\gamma + 2\lambda \geq 0$

$$\begin{aligned}
&\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K(x,y)\,|\varphi(x+y)-\varphi(x)-\varphi(y)|\,\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\
&\leq 4c_1 \int_{\mathbb{R}_*}\int_{(0,y]} x^{-\lambda}y^{\gamma+\lambda}\,|\varphi(x+y)-\varphi(x)-\varphi(y)|\,\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\
&\leq c \int_{\mathbb{R}_*}\int_{(0,y]} x^{-\lambda}y^{\gamma+\lambda}\,(|\varphi(x+y)-\varphi(y)|+|\varphi(x)|)\,\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\
&\leq c||\varphi'||_\infty \int_{\mathbb{R}_*}\int_{(0,1)} x^{1-\lambda}y^{\gamma+\lambda}\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\
&\quad + c \int_{\mathbb{R}_*}\int_{[1,y]} x^{-\lambda}y^{\gamma+\lambda}\,|\varphi(x+y)-\varphi(y)|\,\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\
&\quad + c \int_{\mathbb{R}_*} y^{\gamma+\lambda}\Phi(\mathrm{d}y)\int_{\mathbb{R}_*} x^{-\lambda}|\varphi(x)|\Phi(\mathrm{d}x).
\end{aligned}$$

Using (3.2), the fact that $-\lambda \leq \gamma + \lambda$, as well as the fact that φ is compactly supported the desired conclusion follows.

Theorem 3.3. (Existence of the self-similar profiles) *Let K be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (1.5)*,* (1.6)*, with γ, λ such that* (1.7) *holds and such that*

$$-1 < \gamma, \quad \gamma + 2\lambda \geq 1.$$

Then there exists a self-similar profile Φ as in Definition 3.1. Moreover, Φ is such that $\Phi((0, \rho(M_{\gamma+\lambda}))) = 0$ for

$$\rho(M_{\gamma+\lambda}) := \left(\frac{1-\gamma}{2\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(\mathrm{d}x)}\right)^{\frac{1}{1-\gamma-\lambda}}. \tag{3.4}$$

Additionally, Φ it is such that

$$\int_{\mathbb{R}_*} e^{Lx}\Phi(\mathrm{d}x) < \infty$$

for some $L > 0$ and it is absolutely continuous with respect to the Lebesgue measure. Then $\Phi(\mathrm{d}x) = \phi(x)\mathrm{d}x$ and the density ϕ is such that

$$\limsup_{x\in\mathbb{R}_*} \phi(x)e^{Mx} < \infty$$

for a positive constant M.

Remark 3.4. In this paper we do not prove the uniqueness of the self-similar profiles. Therefore, it makes sense to understand if the proven properties for the self-similar profile constructed in Theorem 5.1 hold for each self-similar profile as in Definition 3.1.

When $\gamma + 2\lambda > 1$ we prove that each self-similar profile as in Definition 3.1 is zero in the set $(0, \rho(M_{\gamma+\lambda}))$, where $\rho(M_{\gamma+\lambda})$ is given by (3.4), see Theorem 6.1 for more details. In contrast, when $\gamma + 2\lambda = 1$, we only prove that the self-similar profile constructed in the proof of Theorem 3.3 is such that $\Phi((0, \rho(M_{\gamma+\lambda}))) = 0$. However, we do not know if this property holds for every self-similar profile as in Definition 3.1.

Theorem 3.5. (Non-existence of the self-similar profiles) *Let K be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (1.5), (1.6)*, with γ, λ such that* (1.7) *holds.*

1. *If $\gamma \leq -1$, $\gamma + 2\lambda > 1$, then a self-similar profile Φ as in Definition* 3.1 *does not exist.*
2. *If $\gamma \leq -1$, $\gamma + 2\lambda = 1$, then a self-similar profile Φ as in Definition* 3.1 *with the additional property*

$$\int_{(0,1]} x^{-\lambda} \Phi(\mathrm{d}x) < \infty \tag{3.5}$$

does not exist.

Remark 3.6. Notice that if $\gamma + 2\lambda > 1$ we prove that self-similar solutions as in Definition 3.1 do not exist when $\gamma \leq -1$. Instead, when $\gamma + 2\lambda = 1$ and $\gamma \leq -1$, we do not exclude the existence of a self-similar solution Φ as in Definition 3.1 with

$$\int_{(0,1]} x^{-\lambda} \Phi(\mathrm{d}x) = \infty.$$

4. Main ideas of the proofs

In this section we explain the main ideas for the proofs of existence/non-existence of self-similar solutions. Both in the case $\gamma + 2\lambda > 1$ and $\gamma + 2\lambda = 1$, to prove that a self-similar solution exists, we find an invariant region for the evolution equation corresponding to (3.1), namely the following equation:

$$\partial_t \Phi(t, \xi) - \frac{3+\gamma}{1-\gamma} \Phi(t, \xi) - \frac{2\xi}{1-\gamma} \partial_\xi \Phi(t, \xi) + \frac{\partial_\xi \left(\xi^{\gamma+\lambda} \Phi(t, \xi) \right)}{\int_{\mathbb{R}_*} x^{\gamma+\lambda} \Phi(t, \mathrm{d}x)} = \mathbb{K}[\Phi](t, \xi). \tag{4.1}$$

By Tychonoff fixed point theorem the existence of an invariant region implies that there exists a solution of Eq. (3.1).

We prove that the set

$$P = \left\{ H : \begin{array}{l} \int_0^\infty x H(x) \mathrm{d}x = 1, \ \frac{1}{C_2} \leq \int_0^\infty x^{\gamma+\lambda} H(x) \mathrm{d}x \leq C_1, \\ H((0, \rho(C_1))) = 0, \ \int_0^\infty x^{2-\gamma-\lambda} H(x) \mathrm{d}x \leq C_2 \end{array} \right\}$$

is invariant when $\gamma + 2\lambda \geq 1$ and $\gamma > -1$ for suitable constants $C_1, C_2 > 0$ and $\rho(C_1)$ given by

$$\rho(C_1) := \left(\frac{1-\gamma}{2C_1}\right)^{\frac{1}{1-\gamma-\lambda}}. \tag{4.2}$$

To prove that the set P is invariant we proceed as follows:

1. We prove that $\int_0^\infty x\Phi_0(x)\mathrm{d}x = 1$ implies $\int_0^\infty x\Phi(t,x)\mathrm{d}x = 1$, for every $t > 0$. This is done multiplying by x Eq. (4.1), integrating from zero to infinity and then studying the ODE for the first order moment obtained in this manner.
2. As a second step we prove that there exists an upper bound for $M_{\gamma+\lambda}$. To this end we multiply both sides of Eq. (3.1) by $x^{\gamma+\lambda}$ and we integrate over x in $(0,\infty)$ to obtain an ODE for the $\gamma+\lambda$ moment. Using Grönwall's lemma, the fact that $\gamma + 2\lambda \geq 1$ and that $\gamma > -1$, the desired conclusion follows.
3. As a third step we use the fact that $M_{\gamma+\lambda} \leq C_1$ to prove that $\Phi(t,(0,\rho(C_1)) = 0$. Indeed, the evolution described by Eq. (4.1) is driven by two mechanisms: coagulation, which increases the average size of the particles in the system, and the growth term,
$$\frac{\xi^{\gamma+\lambda}}{\int_0^\infty x^{\gamma+\lambda}\Phi(t,x)\mathrm{d}x} - \frac{2}{1-\gamma}\xi,$$
which is positive for every $\xi < \rho(C_1)$. Hence, if we start from an initial data Φ_0 such that $\Phi_0((0,\rho(C_1))) = 0$, then we will have that $\Phi(t,(0,\rho(C_1))) = 0$, for every $t > 0$.
4. Using the fact that $\gamma + \lambda < 1$, $\gamma > -1$, $\Phi(t,(0,\rho(C_1))) = 0$, as well as the fact that $\int_0^\infty x^{\gamma+\lambda}\Phi(t,\mathrm{d}x) \leq C_1$, we prove that $\int_0^\infty x^{2-\gamma-\lambda}\Phi(t,x)\mathrm{d}x \leq C_2$.
5. Finally, from the upper bound for $M_{2-\gamma-\lambda}$, we derive a lower bound for $M_{\gamma+\lambda}$. Indeed, Cauchy-Schwarz inequality implies that
$$1 = \left(\int_{\mathbb{R}_*} x\Phi(t,x)\mathrm{d}x\right)^2 \leq \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,x)\mathrm{d}x \int_{\mathbb{R}_*} x^{2-\gamma-\lambda}\Phi(t,x)\mathrm{d}x. \tag{4.3}$$
Hence $\frac{1}{C_2} \leq M_{\gamma+\lambda}$.

In order to prove non-existence we proceed by contradiction. Due to the contribution of the coagulation operator in Eq. (1.16), we expect the zeroth moment of f to decay in time. However, assuming the self-similar change of variable (1.18), we have that

$$\int_0^\infty f(t,x)\mathrm{d}x = t^{-\frac{1+\gamma}{1-\gamma}} \int_0^\infty \Phi(x)\mathrm{d}x.$$

If $\gamma \leq -1$, then $\int_0^\infty f(t,x)\mathrm{d}x$ is constant or increasing in time and this gives a contradiction. Hence we cannot expect self-similar solutions to exist. To make the argument rigorous we will have to prove that $0 < \int_0^\infty \Phi(\mathrm{d}x) < \infty$. When $\gamma + 2\lambda > 1$ we do this by proving that for each self-similar profile there exists a $\delta > 0$ such that $\Phi((0,\delta)) = 0$ and that Φ tends to zero sufficiently fast as $x \to \infty$. Instead, when $\gamma + 2\lambda = 1$, the methods used in this paper do not allow to prove that Φ is equal to zero near the origin, hence we use (3.5) as well as (3.2) to prove that $0 < \int_0^\infty \Phi(\mathrm{d}x) < \infty$.

5. Existence of a self-similar profile

We aim to prove the existence of a solution of Eq. (4.1). Namely, we will prove the following theorem, which is just a reformulation of Theorem 3.3:

Theorem 5.1. *Let K be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying (1.5), (1.6), with γ, λ such that (1.7) holds and such that*

$$-1 < \gamma, \quad \gamma + 2\lambda \geq 1.$$

Then there exists a self-similar profile Φ as in Definition 3.1. Moreover, Φ is such that $\Phi((0, \rho(M_{\gamma+\lambda}))) = 0$ with $\rho(M_{\gamma+\lambda})$ given by (3.4).

To this end we prove the existence of a (time-dependent) solution for a suitably truncated and regularized version of Eq. (4.1):

$$\partial_t \Phi(t,\xi) - \frac{3+\gamma}{1-\gamma}\Phi(t,\xi) - \frac{2\xi}{1-\gamma}\partial_\xi \Phi(t,\xi) + \frac{\partial_\xi\left(\xi^{\gamma+\lambda}\Phi(t,\xi)\right)}{\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,dx)} = \mathbb{K}_R[\Phi](t,\xi). \tag{5.1}$$

The operator $\mathbb{K}_R$ is the truncated operator of parameter $R > 0$ defined as

$$\begin{aligned}\mathbb{K}_R[\Phi](t,x) :=& \frac{1}{2}\int_0^x K_R(x-y,y)\,\Phi(t,x-y)\,\Phi(t,y)\,\mathrm{d}y \\ &- \int_0^\infty K_R(x,y)\,\Phi(t,x)\,\Phi(t,y)\,\mathrm{d}y,\end{aligned} \tag{5.2}$$

where the kernel K_R is a *truncated kernel*, i.e. it is a continuously differentiable, bounded and symmetric function $K_R : \mathbb{R}_*^2 \to \mathbb{R}_+$, such that for every $(x, y) \in \mathbb{R}_*^2$ we have that $K_R(x, y) \leq K(x, y)$ and

$$\begin{cases} K_R(x,y) = 0, & \text{if } x > R \text{ or } y > R; \\ |K_R(x,y) - K(x,y)| \leq e^{-R}, & \text{if } (x,y) \in \left[\frac{1}{2R}, \frac{R}{4}\right]^2. \end{cases} \tag{5.3}$$

Notice that the kernel K_R can be obtained starting from the kernel K by means of standard truncations and mollifying arguments.

Using Tychonoff fixed point theorem we prove the existence of a stationary solution Φ_R for Eq. (5.1) and we will prove that there exists the limit Φ of $\{\Phi_R\}_R$ as R tends to infinity. To conclude we will prove that the measure Φ satisfies Eq. (1.19).

5.1. Existence of a time dependent solution for the truncated equation

Since in this section we work only with the truncated equation, we omit the label R in Φ_R. We will reintroduce the label R in Sect. 5.3. We start this section by introducing a definition of solutions for Eq. (5.1).

Definition 5.2. Let K_R be the truncated kernel defined as in (5.3) as a function of the homogeneous symmetric coagulation kernel K satisfying (1.5), (1.6), with homogeneity $\gamma < 1$ and with $\gamma + \lambda < 1$ and $\gamma + 2\lambda \geq 1$. A function $\Phi \in C^1([0,T];\mathcal{M}_{+,b}(\mathbb{R}_*))$ is a solution of Eq. (5.1) if

$$0 < \inf_{t\in[0,T]} \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,\mathrm{d}x) \ \text{ and } \ \sup_{t\in[0,T]} \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,\mathrm{d}x) < \infty$$

and if Φ satisfies

$$\begin{aligned}
&\int_{\mathbb{R}_*} \varphi(x)\dot{\Phi}(t,\mathrm{d}x) - \frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*} \varphi(x)\Phi(t,\mathrm{d}x) + \frac{2}{1-\gamma}\int_{\mathbb{R}_*} \varphi'(x)x\Phi(t,\mathrm{d}x) \\
&\quad - \frac{1}{\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t,\mathrm{d}x)}\int_{\mathbb{R}_*} \varphi'(x)x^{\gamma+\lambda}\Phi(t,\mathrm{d}x) \\
&\quad = \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_R(x,y)\left[\varphi(x+y)-\varphi(x)-\varphi(y)\right]\Phi(t,\mathrm{d}x)\Phi(t,\mathrm{d}y),
\end{aligned} \tag{5.4}$$

for every test function $\varphi \in C_c^1(\mathbb{R}_*)$.

Given the positive numbers k_1, k_2, ρ^*, we define the subset $S(k_1,k_2,\rho^*)$ of $\mathcal{M}_+(\mathbb{R}_*)$ as

$$S(k_1,k_2,\rho^*) := \left\{ H \in \mathcal{M}_+(\mathbb{R}_*) : \begin{array}{l} H((0,\rho^*)) = 0, \int_{\mathbb{R}_*} xH(\mathrm{d}x) = 1, \\ k_1 \leq \int_{\mathbb{R}_*} x^{\gamma+\lambda}H(\mathrm{d}x) \leq k_2 \end{array} \right\}. \tag{5.5}$$

Theorem 5.3. *Assume K_R to be a truncated kernel defined as in (5.3) as a function of a homogeneous symmetric kernel K satisfying (1.5), (1.6), with parameters $\gamma, \lambda \in \mathbb{R}$ such that (1.7) holds and $\gamma + 2\lambda \geq 1$ and $\gamma > -1$. Then for all $k_1, k_2 > 0$ and sufficiently large $R_0 > 0$ such that $\Phi_0 \in S\left(2k_1, \frac{k_2}{2}, \rho(k_2)\right) \cap \{H \in \mathcal{M}_+(\mathbb{R}_*) : H((4R,\infty)) = 0\}$ for $R > R_0$ and for $\rho(k_2)$ given by (4.2), there exists a unique solution $\Phi \in C^1([0,T];\mathcal{M}_+(\mathbb{R}_*))$ of Eq. (5.1) in the sense of Definition 5.2, for $T > 0$ sufficiently small. For every $t \in [0,T]$, we have that $\Phi(t,\cdot) \in S(k_1,k_2,\rho(k_2)) \cap \{H \in \mathcal{M}_+(\mathbb{R}_*) : H((4R,\infty)) = 0\}$.*

Lemma 5.4. *Let $T > 0$ and let $R > 0$. Assume $\gamma < 1$ and $\gamma + \lambda < 1$. Assume $\alpha \in C([0,T])$ satisfying*

$$0 < k_1 \leq \alpha(t) \leq k_2 < \infty, \quad \textit{for all } \ t \in [0,T], \tag{5.6}$$

for some positive constants k_1 and k_2. Consider the ODE

$$\frac{dx(t)}{dt} = V(x(t),\alpha(t)), \quad x(0) = x_0 > 0, \tag{5.7}$$

with $V(x,\alpha) := \frac{x^{\gamma+\lambda}}{\alpha} - \frac{2}{1-\gamma}x$*. If* $x_0 \geq \rho(k_2)$ *with* $\rho(k_2)$ *given by* (4.2)*, then* (5.7) *has a unique solution* $X(t, x_0, \alpha) \geq \rho(k_2)$*. Let* $X(t, x_{0,1}, \alpha_1)$ *and* $X(t, x_{0,2}, \alpha_2)$ *be the solutions of Eq.* (5.7) *with respect to the functions* α_1 *and* α_2 *satisfying* (5.6) *and* $x_{0,1}, x_{0,2} \in [\rho(k_2), 4R]$*. Then*

$$|X(t, x_{0,1}, \alpha_1) - X(t, x_{0,2}, \alpha_2)| \leq L_1|x_{0,1} - x_{0,2}| + TL_2\|\alpha_1 - \alpha_2\|_{[0,T]}, \tag{5.8}$$

for every $t \in [0, T]$*, where* $L_1 = L_1(T, \gamma, \lambda, k_1, k_2, R) > 0$*, and* $L_2 = L_2(T, \gamma, \lambda, k_1, k_2, R) > 0$ *and where we denote by* $\|\cdot\|_{[0,T]}$ *the norm* $\|f\|_{[0,T]} := \sup_{t\in[0,T]}|f(t)|$*, for* $f \in C([0, T])$*.*

Proof. Since $V(x,\alpha) > 0$, for every $x \leq \rho(k_2)$, we deduce that the set $\{x \geq \rho(k_2)\}$ is an invariant region of (5.7). This also implies that a unique solution exists.

Equation (5.7) is a Bernoulli equation that can be reduced to the linear ODE

$$\frac{dy}{dt} = \frac{1-(\gamma+\lambda)}{\alpha(t)} - \frac{2(1-(\gamma+\lambda))}{1-\gamma}y, \quad y_0 := x_0^{1-(\gamma+\lambda)}$$

via the change of variable $y = x^{1-(\lambda+\gamma)}$. We deduce that

$$Y(t, x_0^{1-(\gamma+\lambda)}, \alpha) = x_0^{1-(\gamma+\lambda)} e^{-\frac{2(1-(\gamma+\lambda))}{1-\gamma}t} + \int_0^t \frac{(1-\gamma-\lambda)}{\alpha(s)} e^{-\frac{2(1-(\gamma+\lambda))}{1-\gamma}(t-s)} \mathrm{d}s.$$

Since α_1, α_2 satisfy (5.6) and since $x_{0,1}, x_{0,2} \in [\rho(k_2), 4R]$, the above formula implies that

$$\left|Y(t, x_{0,1}^{1-(\gamma+\lambda)}, \alpha_1) - Y(t, x_{0,2}^{1-(\gamma+\lambda)}, \alpha_2)\right| \leq \left|x_{0,1}^{1-(\gamma+\lambda)} - x_{0,2}^{1-(\gamma+\lambda)}\right| + C(T, \gamma, \lambda, k_1)\|\alpha_1 - \alpha_2\|_{[0,T]}.$$

Since the set $\{x \geq \rho(k_2)\}$ is invariant for (5.7), the above inequality, together with the definition of y as a function of x, implies (5.8). □

We will solve Eq. (5.1) using Lagrangian coordinates. To this end we introduce the notation

$$\chi_\varphi^\alpha(x, y, t) := \varphi(\ell_\alpha(x, y, t)) - \varphi(x) - \varphi(y),$$

for $\varphi \in C_c(\mathbb{R}_*)$, where ℓ_α is defined for $x > 0$, $y > 0$ and $t \geq 0$ as

$$X(t, \ell_\alpha(x, y, t), \alpha) = X(t, x, \alpha) + X(t, y, \alpha). \tag{5.9}$$

The function ℓ_α is well defined because for every fixed time $t \geq 0$ the function $x \mapsto X(t, x, \alpha)$ is increasing.

Lemma 5.5. *Let K_R be a truncated kernel defined as in (5.3) as a function of a homogeneous symmetric kernel K satisfying (1.5), (1.6), with $\gamma, \lambda \in \mathbb{R}$ satisfying (1.7). Let k_1, k_2, R be three positive constants. Assume that the initial condition Φ_0 is such that $2k_1 \leq \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi_0(\mathrm{d}x) \leq \frac{k_2}{2}$ and that $\Phi_0((0, \rho(k_2)) \cup (4R, \infty)) = 0$ for $\rho(k_2)$ given by (4.2).*

For a sufficiently small time $T > 0$, depending on R, there exists a function $F \in C([0, T], \mathcal{M}_+(\mathbb{R}_*))$, with $F(0, \cdot) = \Phi_0$, that satisfies

$$\int_{\mathbb{R}_*} \varphi(x)F(t, \mathrm{d}x) = \int_{\mathbb{R}_*} \varphi(x)\Phi_0(\mathrm{d}x) + \frac{1}{2}\int_0^t \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_R(X(s, x, \alpha), X(s, y, \alpha))e^{\frac{1+\gamma}{1-\gamma}s}\chi_\varphi^\alpha(x, y, s)F(s, \mathrm{d}x)F(s, \mathrm{d}y)\mathrm{d}s, \tag{5.10}$$

for every $\varphi \in C_c(\mathbb{R}_*)$, with

$$\alpha(t) := e^{\frac{1+\gamma}{1-\gamma}t}\int_{\mathbb{R}_*} (X(t, x, \alpha))^{\gamma+\lambda}\, F(t, \mathrm{d}x), \quad \forall t \in [0, T]. \tag{5.11}$$

The function F is such that $k_1 \leq \alpha(t) \leq k_2$ and such that

$$F(t, (0, \rho(k_2)) \cup (4R, \infty)) = 0, \quad \forall t \in [0, T]. \tag{5.12}$$

Proof. We define the set

$$\tilde{X} := \{H \in \mathcal{M}_+(\mathbb{R}_*) : H((0, \rho(k_2)) \cup (4R, \infty)) = 0\}$$

and the set

$$X_T := \left\{ (F, \alpha) \in C([0, T]; \mathcal{M}_+(\mathbb{R}_*)) \times C([0, T]) : \begin{array}{c} F(t, \cdot) \in \tilde{X}, k_1 \leq \alpha(t) \leq k_2, \forall t \in [0, T], \\ \sup_{t\in[0,T]} \int_{\mathbb{R}_*} F(t, \mathrm{d}x) \leq 1 + \int_{\mathbb{R}_*} \Phi_0(\mathrm{d}x) \end{array} \right\}.$$

We endow the set $\mathcal{M}_+(\mathbb{R}_*)$ with the Wasserstein metric W_1. The reason for this choice will become clear in the proof of the contractivity of the evolution operator corresponding to (5.10), (5.11). We endow the set $C([0, T], \mathcal{M}_+(\mathbb{R}_*))$ with the metric induced by the distance

$$d_T(\mu, \nu) := \sup_{t\in[0,T]} W_1(\mu(t, \cdot), \nu(t, \cdot)).$$

Similarly, we endow $C([0, T])$ with the norm $\|f\|_{[0,T]} := \sup_{t\in[0,T]} |f(t)|$.

For every $(F, \alpha) \in X_T$, we define the operator $\mathcal{T}[F, \alpha](t) : C_c(\mathbb{R}_*) \to \mathbb{R}_+$ as $\mathcal{T}[F, \alpha](t) := \mathcal{T}_1[F, \alpha](t) + \mathcal{T}_2[F, \alpha](t)$ with

$$\langle \mathcal{T}_1[F, \alpha](t), \varphi\rangle = \int_{\mathbb{R}_*} \varphi(x)e^{-\int_0^t a[F,\alpha](s,x)ds}\Phi_0(dx),$$

$$a[F, \alpha](t, x) := e^{\frac{\gamma+1}{1-\gamma}t}\int_0^\infty K_R(X(t, x, \alpha), X(t, y, \alpha))F(t, dy),$$

$$\langle \mathcal{T}_2[F, \alpha](t), \varphi\rangle = \frac{1}{2}\int_0^t \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} e^{-\int_s^t a[F](v,x)dv} K_R(X(s, x, \alpha), X(s, y, \alpha)) \times e^{\frac{1+\gamma}{1-\gamma}s}\varphi(\ell_\alpha(x, y, s)) \cdot F(s, dx)F(s, dy)ds.$$

Moreover, given $(F,\alpha) \in X_T$, we define the operator $\mathcal{A}[F,\alpha] : [0,T] \to \mathbb{R}_*$ as

$$\mathcal{A}[F,\alpha](t) := e^{\frac{1+\gamma}{1-\gamma}t} \int_{\mathbb{R}_*} (X(t,x,\alpha))^{\gamma+\lambda} F(t,\mathrm{d}x), \quad \forall t \in [0,T].$$

We can now rewrite (5.10), (5.11) in a fixed point form, namely as $(F,\alpha) = \mathcal{F}[F,\alpha]$ where $\mathcal{F}[F,\alpha] := (\mathcal{T}[F,\alpha], \mathcal{A}[\mathcal{T}[F,\alpha],\alpha])$.

We prove now that $\mathcal{F} : X_T \to X_T$. Since for every $(F,\alpha) \in X_T$, the operator $\mathcal{T}[F,\alpha](t)$ is linear and continuous, we deduce that it can be identified with an element of $\mathcal{M}_{+,b}(\mathbb{R}_*)$. Moreover, the operator $t \mapsto \mathcal{T}[F,\alpha](t)$ is a continuous map from $\mathbb{R}_+$ to $\mathcal{M}_{+,b}(\mathbb{R}_*)$, hence $\mathcal{T}[F,\alpha] \in C([0,T],\mathcal{M}_{+,b}(\mathbb{R}_*))$. Similarly, for every $(F,\alpha) \in X_T$, we have that $\mathcal{A}[F,\alpha] \in C([0,T])$.

Consider a test function φ such that $\varphi(x) = 0$ for every $x \in [\rho(k_2),\infty)$. Then, since $\Phi_0((0,\rho(k_2))) = 0$, we deduce that

$$\langle \mathcal{T}_1[F,\alpha](t), \varphi\rangle = \int_{[\rho(k_2),\infty)} \varphi(x) e^{-\int_0^t a[F,\alpha](s,x)\mathrm{d}s} \Phi_0(\mathrm{d}x) = 0.$$

Similarly, since $\ell_\alpha(s,x,y) \geq x$ and since $(F,\alpha) \in X_T$, we deduce that

$$\langle \mathcal{T}_2[F,\alpha](t), \varphi\rangle \leq \frac{C(R)}{2} \int_0^t \int_{[\rho(k_2),\infty)} \int_{[\rho(k_2),\infty)} e^{\frac{1+\gamma}{1-\gamma}s} \varphi(\ell_\alpha(x,y,s)) \\ \times F(s,dx)F(s,dy)ds = 0.$$

Therefore, $(\mathcal{T}[F,\alpha])(t,(0,\rho(k_2))) = 0$, for every $t \in [0,T]$.

Consider any test function φ such that $\varphi(x) = 0$ in $[0,4R]$. Since $\Phi_0((4R,\infty)) = 0$, we have that

$$\langle \mathcal{T}_1[F,\alpha](t), \varphi\rangle \leq \int_{(0,4R]} \varphi(x) e^{-\int_0^t a[F,\alpha](s,x)\mathrm{d}s} \Phi_0(\mathrm{d}x) = 0.$$

Moreover, using the notation $S_1(s) := \{(x,y) \in \mathbb{R}_*^2 : X(s,x,\alpha) \leq R, X(s,y,\alpha) \leq R\}$, $S_2(s) := \{(x,y) \in \mathbb{R}_*^2 : X(s,x,\alpha) > R, X(s,y,\alpha) \leq R\}$ and $S_3(s) := \{(x,y) \in \mathbb{R}_*^2 : X(s,x,\alpha) > R, X(s,y,\alpha) > R\}$, we have

$$\begin{aligned}
&\langle \mathcal{T}_2[F,\alpha](t), \varphi\rangle \\
&\quad \leq \frac{1}{2}\int_0^t \iint_{S_1(s)} e^{\frac{1+\gamma}{1-\gamma}s} K_R(X(s,x,\alpha),X(s,y,\alpha))\varphi(\ell_\alpha(x,y,s))F(s,\mathrm{d}x)F(s,\mathrm{d}y)\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^t \iint_{S_2(s)} e^{\frac{1+\gamma}{1-\gamma}s} K_R(X(s,x,\alpha),X(s,y,\alpha))\varphi(\ell_\alpha(x,y,s))F(s,\mathrm{d}x)F(s,\mathrm{d}y)\mathrm{d}s \\
&\qquad + \frac{1}{2}\int_0^t \iint_{S_3(s)} e^{\frac{1+\gamma}{1-\gamma}s} K_R(X(s,x,\alpha),X(s,y,\alpha))\varphi(\ell_\alpha(x,y,s))F(s,\mathrm{d}x)F(s,\mathrm{d}y)\mathrm{d}s.
\end{aligned}$$

For x, y such that $X(s,x,\alpha) \geq R$ or $X(s,y,\alpha) \geq R$, we have that $K_R(X(s,x,\alpha), X(s,y,\alpha)) = 0$. Thus, the second and the third terms above are equal to zero.

To see that the first term is equal to zero, notice that $X(t,x,\alpha) \geq xe^{-\frac{2}{1-\gamma}t}$. We can thus select T sufficiently small so that $X(t,x,\alpha) \geq \frac{x}{2}$. This implies that

$\frac{1}{2}\ell_\alpha(x, y, s) \leq X(s, \ell_\alpha(x, y, s), \alpha) = X(s, x, \alpha) + X(s, y, \alpha) \leq 2R$. Hence $\ell_\alpha(x, y, s) \leq 4R$. Since $\varphi(x) = 0$ for every $x \leq 4R$, the desired conclusion follows.

We now prove that $k_1 \leq \mathcal{A}[\mathcal{T}[F, \alpha], \alpha](t) \leq k_2$. To this end notice that

$$\begin{aligned}
&\left|\mathcal{A}[\mathcal{T}[F, \alpha], \alpha](t) - \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi_0(\mathrm{d}x)\right| \\
&\leq \int_{\mathbb{R}_*} \left|e^{\frac{1+\gamma}{1-\gamma}t}(X(t, x, \alpha))^{\gamma+\lambda} - x^{\gamma+\lambda}\right| \mathcal{T}[F, \alpha](t, \mathrm{d}x) \\
&\quad + \int_{\mathbb{R}_*} x^{\gamma+\lambda} \left|\mathcal{T}[F, \alpha](t, \mathrm{d}x) - \Phi_0(\mathrm{d}x)\right|.
\end{aligned}$$

Notice that

$$\begin{aligned}
\left|e^{\frac{1+\gamma}{1-\gamma}t}(X(t, x, \alpha))^{\gamma+\lambda} - x^{\gamma+\lambda}\right| &\leq e^{\frac{1+\gamma}{1-\gamma}t}\left|(X(t, x, \alpha))^{\gamma+\lambda} - x^{\gamma+\lambda}\right| \\
&\quad + \frac{1+\gamma}{1-\gamma} T x^{\gamma+\lambda} \\
&\leq TC(T, R, k_2).
\end{aligned}$$

Finally, we have that the term

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda} \left|\mathcal{T}[F, \alpha](t, \mathrm{d}x) - \Phi_0(\mathrm{d}x)\right|$$

can be arbitrarily small by taking T small. This is due to the definition of the map $T[F, \alpha]$ and the upper bound for F in the definition of the space X_T.

We deduce that, for small time T, we have that $\left|\mathcal{A}[F, \alpha](t) - \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi_0(\mathrm{d}x)\right| \leq C(T)$, for every $t \in [0, T]$, where the constant $C(T)$ can be made arbitrarily small.

We now check that the map $\mathcal{F}$ is a contraction. To this end, we use the fact that $|e^{-x_1} - e^{-x_2}| \leq |x_1 - x_2|$, for $x_1, x_2 \geq 0$. Consider $(F, \alpha), (G, \beta) \in X_T$. Then, for every Lipschitz function φ with $||\varphi||_{\mathrm{Lip}} \leq 1$, we have that

$$\begin{aligned}
&\left|\int_{\mathbb{R}_*} \varphi(x)[\mathcal{T}_1[F, \alpha] - \mathcal{T}_1[G, \beta]](t, \mathrm{d}x)\right| \\
&\leq \int_{\mathbb{R}_*}\int_0^t |\varphi(x)\left[a[F, \alpha](s, x) - a[G, \beta](s, x)\right]| \,\mathrm{d}s\,\Phi_0(\mathrm{d}x) \\
&\leq \int_{\mathbb{R}_*}\int_0^t \left|\varphi(x)\left[\int_{\mathbb{R}_*} K_R(X(s, x, \alpha), X(s, y, \alpha))F(s, \mathrm{d}y)\right.\right. \\
&\quad \left.\left. - \int_{\mathbb{R}_*} K_R(X(s, x, \beta), X(s, y, \beta))G(s, \mathrm{d}y)\right] \mathrm{d}s\,\Phi_0(\mathrm{d}x)\right| \\
&\leq I_1[\varphi] + I_2[\varphi],
\end{aligned}$$

where

$$I_1[\varphi] := \int_{\mathbb{R}_*}\int_0^t \Big|\varphi(x)\int_{\mathbb{R}_*}[K_R(X(s,x,\alpha),X(s,y,\alpha)) - K_R(X(s,x,\beta),X(s,y,\beta))]F(s,\mathrm{d}y)\Big|\mathrm{d}s\,\Phi_0(\mathrm{d}x)$$

and

$$I_2[\varphi] := \int_{\mathbb{R}_*}\int_0^t \Big|\varphi(x)\int_{\mathbb{R}_*} K_R(X(s,x,\beta),X(s,y,\beta))\,(F(s,\mathrm{d}y)-G(s,\mathrm{d}y))\,\mathrm{d}s\,\Phi_0(\mathrm{d}x)\Big|.$$

Then, the differentiability of the kernel K_R, together with inequality (5.8), implies that

$$\sup_{||\varphi||_{\mathrm{Lip}\le 1}} I_1[\varphi] \le C(R)T\|\Phi_0\|_{TV}(1+\|\Phi_0\|_{TV})\|\alpha-\beta\|_{[0,T]}.$$

Similarly,

$$\sup_{||\varphi||_{\mathrm{Lip}\le 1}} I_2[\varphi] \le C(R)T\|\Phi_0\|_{TV} d_T(F,G).$$

Using again the differentiability of the kernel K_R, inequality (5.8) and the definition of ℓ_α, we obtain that

$$\begin{aligned}\sup_{||\varphi||_{\mathrm{Lip}\le 1}} &\left|\int_{\mathbb{R}_*}\varphi(x)[\mathcal{T}_2[F,\alpha](t,\mathrm{d}x)-\mathcal{T}_2[G,\beta](t,\mathrm{d}x)]\right| \\ &\le C(T,R)(1+||\Phi_0||_{TV})^2 d_T(F,G) \\ &\quad + C(T,R)(1+||\Phi_0||_{TV})^2\|\alpha-\beta\|_{[0,T]},\end{aligned}$$

where T can be selected so that $C(T,R)(1+||\Phi_0||_{TV})^2 \le \frac{1}{4}$.

We remark that, in order to obtain this bound, we used the properties of the Wasserstain distance. Similar estimates cannot be obtained by endowing $\mathcal{M}_+(\mathbb{R}_*)$ with the total variation norm.

To understand this better, let $\alpha,\beta\in C([0,T])$, with $k_1\le\alpha,\beta\le k_2$. Let $s\in[0,T]$. Using the fact that $||\varphi||_{\mathrm{Lip}}\le 1$ and that we can assume we work on a sufficiently nice compact set due to the support of the measures, we can derive the estimate

$$\begin{aligned}|\varphi(\ell_\alpha(x,y,s))-\varphi(\ell_\beta(x,y,s))| &\le |\ell_\alpha(x,y,s)-\ell_\beta(x,y,s)| \\ &\lesssim |X(s,\ell_\alpha(x,y,s),\alpha)-X(s,\ell_\beta(x,y,s),\alpha)| \\ &\le |X(s,\ell_\alpha(x,y,s),\alpha)-X(s,\ell_\beta(x,y,s),\beta)| \\ &\quad + |X(s,\ell_\beta(x,y,s),\beta)-X(s,\ell_\beta(x,y,s),\alpha)|.\end{aligned}$$

Using (5.8) and (5.9), we have

$$\begin{aligned}|X(s,\ell_\alpha(x,y,s),\alpha)-X(s,\ell_\beta(x,y,s),\beta)| &\le |X(s,x,\alpha)-X(s,x,\beta)| \\ &\quad + |X(s,y,\alpha)-X(s,y,\beta)| \\ &\le 2TL_2||\alpha-\beta||_{[0,T]}\end{aligned}$$

and thus

$$|\varphi(\ell_\alpha(x,y,s)) - \varphi(\ell_\beta(x,y,s))| \lesssim T||\alpha-\beta||_{[0,T]}. \tag{5.13}$$

More precisely, inequality (5.13) was needed to prove the contractivity of the map $\mathcal{T}_2$.

In order to prove the upper bound for the map $T[F,\alpha]$, we first observe that

$$\langle \mathcal{T}_1[F,\alpha](t),1\rangle = \int_{[\rho(k_2),\infty)} e^{-\int_0^t a[F,\alpha](s,x)\mathrm{d}s}\Phi_0(\mathrm{d}x) \leq ||\Phi_0||_{TV}$$

and then that

$$\begin{aligned}\langle \mathcal{T}_2[F,\alpha](t),1\rangle &\leq \frac{C(R)}{2}e^{\frac{1+\gamma}{1-\gamma}t}\int_0^t\int_{[\rho(k_2),\infty)}\int_{[\rho(k_2),\infty)} F(s,\mathrm{d}x)F(s,\mathrm{d}y)\mathrm{d}s\\ &\leq \frac{C(R)}{2}e^{\frac{1+\gamma}{1-\gamma}T}T(1+||\Phi_0||_{TV})^2 \leq 1,\end{aligned}$$

for T sufficiently small. Thus,

$$\sup_{t\in[0,T]}\langle \mathcal{T}[F,\alpha](t),1\rangle \leq 1+||\Phi_0||_{TV}.$$

Finally,

$$\begin{aligned}&\|\mathcal{A}[\mathcal{T}[F,\alpha],\alpha]-\mathcal{A}[\mathcal{T}[G,\beta],\beta]\|_{[0,T]}\\ &\quad\leq C(R)d_T([\mathcal{T}[F,\alpha],\mathcal{T}[G,\beta])\\ &\qquad+C_1(T,R)T(1+||\Phi_0||_{TV})\|\alpha-\beta\|_{[0,T]}\\ &\quad\leq C(T,R)T(1+||\Phi_0||_{TV})^2[d_T(F,G)+\|\alpha-\beta\|_{[0,T]}],\end{aligned}$$

where the constant $TC(T,R)(1+||\Phi_0||_{TV})^2$ can be made smaller than $\frac{1}{4}$ by selecting T small. The operator $\mathcal{F}$ is therefore a contraction and, by Banach fixed point theorem, we conclude that it has a unique fixed point $(F,\alpha)\in X_T$. □

Proof of Theorem 5.3. Assume $T>0$ to be as in Lemma 5.5. Let $(F,\alpha)\in C([0,T],\mathcal{M}_+(\mathbb{R}_*))\times C([0,T])$ be the solution of (5.10), (5.11). Let $X(t,y,\alpha)$ be the solution of Eq. (5.7). Let $\Phi\in C^1([0,T],\mathcal{M}_+(\mathbb{R}_*))$ be the function defined by duality as

$$\int_{\mathbb{R}_*}\varphi(x)\Phi(t,\mathrm{d}x) = \int_{\mathbb{R}_*}\varphi(X(t,y,\alpha))e^{\frac{1+\gamma}{1-\gamma}t}F(t,\mathrm{d}y), \tag{5.14}$$

for every $\varphi\in C_c(\mathbb{R}_*)$. We prove that the function Φ is such that

$$\Phi(t,(0,\rho(k_2))\cup(4R,\infty)) = 0, \quad \forall t\in[0,T], \tag{5.15}$$

for $\rho(k_2)$ defined as in (4.2). To this end notice that, since $F(t,(0,\rho(k_2))=0$, for every $t\in[0,T]$, and since $X(t,y,\alpha)\geq\rho(k_2)$, for every $y\geq\rho(k_2)$, we have that

$$\int_{\mathbb{R}_*}\varphi(x)\Phi(t,\mathrm{d}x) = \int_{[\rho(k_2),\infty)}\varphi(X(t,y,\alpha))e^{\frac{1+\gamma}{1-\gamma}t}F(t,\mathrm{d}y) = 0,$$

for every test function φ such that $\varphi(x) = 0$ if $x \geq \rho(k_2)$. Similarly, since $t \mapsto X(t, x, \alpha)$ is a decreasing function for large values of x, we deduce that $X(t, x, \alpha) \leq 4R$, for every $x \leq 4R$. Hence,

$$\int_{\mathbb{R}_*} \varphi(x)\Phi(t, \mathrm{d}x) = \int_{[\rho(k_2),4R]} \varphi(X(t, y, \alpha))e^{\frac{1+\gamma}{1-\gamma}t} F(t, \mathrm{d}y) = 0,$$

for every test function φ such that $\varphi(x) = 0$ if $x \leq 4R$.

We now prove that Φ satisfies (5.4). By its definition as the fixed point of the operator $\mathcal{F}$, we deduce that $F \in C^1([0, T], \mathcal{M}_+(\mathbb{R}_*))$. Differentiating both sides of the equality (5.14) in time, we deduce that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*} \varphi(x)\Phi(t, \mathrm{d}x) &= \int_{\mathbb{R}_*} \varphi(X(t, y, \alpha))e^{\frac{1+\gamma}{1-\gamma}t} \partial_t F(t, \mathrm{d}y)\\ &\quad + \int_{\mathbb{R}_*} \varphi'(X(t, y, \alpha))\Bigg[\frac{(X(t, y, \alpha))^{\gamma+\lambda}}{\alpha(t)}\\ &\quad - \frac{2}{1-\gamma}X(t, y, \alpha)\Bigg]e^{\frac{1+\gamma}{1-\gamma}t} F(t, \mathrm{d}y)\\ &\quad + \frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*} \varphi(X(t, y, \alpha))e^{\frac{1+\gamma}{1-\gamma}t} F(t, \mathrm{d}y).\end{aligned}$$

Using the fact that F satisfies Eq. (5.10), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}_*} \varphi(X(t, y, \alpha))e^{\frac{1+\gamma}{1-\gamma}t} \partial_t F(t, \mathrm{d}y)\\ &\quad = \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} e^{\frac{2(1+\gamma)}{1-\gamma}t} F(t, \mathrm{d}x)F(t, \mathrm{d}y)K_R(X(t, x, \alpha), X(t, y, \alpha))\\ &\quad\quad \times[\varphi(X(t, x, \alpha) + X(t, y, \alpha)) - \varphi(X(t, y, \alpha)) - \varphi(X(t, x, \alpha))].\end{aligned}$$

Hence, using again the equality (5.14), we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}_*} \varphi(x)\Phi(t, \mathrm{d}x) &= \int_{\mathbb{R}_*} \varphi'(X(t, y, \alpha))\Bigg[\frac{(X(t, y, \alpha))^{\gamma+\lambda}}{\alpha(t)}\\ &\quad - \frac{2}{1-\gamma}X(t, y, \alpha)\Bigg]e^{\frac{1+\gamma}{1-\gamma}t} F(t, \mathrm{d}y)\\ &\quad + \frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*} \varphi(X(t, y, \alpha))e^{\frac{1+\gamma}{1-\gamma}t} F(t, \mathrm{d}y)\\ &\quad + \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_R(X(t, x, \alpha), X(t, y, \alpha))e^{\frac{2(1+\gamma)}{1-\gamma}t} F(t, \mathrm{d}x)F(t, \mathrm{d}y)\\ &\quad \cdot [\varphi(X(t, x, \alpha) + X(t, y, \alpha)) - \varphi(X(t, y, \alpha)) - \varphi(X(t, x, \alpha))]\\ &= \int_{\mathbb{R}_*} \varphi'(y)\left[\frac{y^{\gamma+\lambda}}{M_{\gamma+\lambda}(\Phi(t))} - \frac{2}{1-\gamma}y\right]\Phi(t, \mathrm{d}y)\\ &\quad + \frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*} \varphi(y)\Phi(t, \mathrm{d}y)\\ &\quad + \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_R(x, y)\,[\varphi(x + y) - \varphi(y)\\ &\quad - \varphi(x)]\,\Phi(t, \mathrm{d}x)\Phi(t, \mathrm{d}y).\end{aligned}$$

We conclude that Φ satisfies (5.4). □

5.2. Existence of a stationary solution for the truncated equation

Theorem 5.6. *Assume K_R to be a truncated kernel as in (5.3) defined as a function of the homogeneous symmetric kernel K that satisfies (1.5), (1.6), for parameters $\gamma, \lambda \in \mathbb{R}$ such that (1.7) holds and such that $\gamma + 2\lambda \geq 1$, $\underline{\gamma} > -1$. There exists $\overline{R} > 0$ such that, for every truncation parameter $R > \overline{R}$, there exists a $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ satisfying the equation*

$$\int_{\mathbb{R}_*} \varphi'(x) \left[\frac{2}{1-\gamma} x - \frac{x^{\gamma+\lambda}}{\int_{\mathbb{R}_*} z^{\gamma+\lambda} \Phi(\mathrm{d}z)} \right] \Phi(\mathrm{d}x) - \frac{1+\gamma}{1-\gamma} \int_{\mathbb{R}_*} \varphi(x) \Phi(\mathrm{d}x)$$
$$= \frac{1}{2} \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K_R(x, y) \left[\varphi(x+y) - \varphi(x) - \varphi(y)\right] \Phi(\mathrm{d}x)\Phi(\mathrm{d}y), \tag{5.16}$$

for every $\varphi \in C_c^1(\mathbb{R}_*)$. The solution is such that

$$\int_{\mathbb{R}_*} x\Phi(\mathrm{d}x) = 1, \quad \int_{\mathbb{R}_*} x^{\gamma+\lambda} \Phi(\mathrm{d}x) \leq C_1 \quad \text{and} \quad \int_{\mathbb{R}_*} x^{2-\gamma-\lambda} \Phi(\mathrm{d}x) \leq C_2, \tag{5.17}$$

for some constants $C_1, C_2 > 0$ that do not depend on the truncation R. Additionally, Φ is such that $\Phi((0, \rho(C_1))) = 0$ where $\rho(C_1)$ is given by (4.2).

To prove the theorem we introduce the semigroup $\{S(t)\}_{\{t \geq 0\}}$ defined as $S(0)\Phi_0 = \Phi_0$ and $S(t)\Phi_0 = \Phi(t, \cdot)$, where Φ is the solution of Eq. (5.4) constructed in the previous section. We want to apply Tychonoff fixed point theorem to prove the existence of a stationary solution for Eq. (5.1). To this end we need to find an invariant region for $S(t)$ (Proposition 5.7) and prove the continuity in the weak-* topology of the map $\Phi \mapsto S(t)\Phi$ for every time t (Proposition 5.12). In this way we prove that for every time t there exists a fixed point $\hat{\Phi}_t$ such that $S(t)\hat{\Phi}_t = \hat{\Phi}_t$. We will then prove that S is continuous (Proposition 5.8) in time and conclude that the limit as t tends to zero of $\hat{\Phi}_t$ is a solution of Eq. (5.16). This is a standard method used in the study of coagulation equations to prove existence of self-similar profiles. For more details, see [8, Theorem 1.2].

Proposition 5.7. (Invariant region) *Assume K_R to be a truncated kernel as in (5.3) defined as a function of the homogeneous symmetric kernel K that satisfies (1.5), (1.6), for parameters $\gamma, \lambda \in \mathbb{R}$ such that (1.7) holds and such that $\gamma + 2\lambda \geq 1$, $\gamma > -1$. Then there exist some positive constants C_1, C_2 that do not depend on the truncation R such that the set*

$$P := \left\{ H \in \mathcal{M}_+(\mathbb{R}_*) : \begin{array}{l} M_1(H) = 1, \ H((0, \rho(C_1)) \cup (4R, \infty)) = 0, \\ \dfrac{1}{C_2} \leq M_{\gamma+\lambda}(H) \leq C_1 \end{array} \right\} \tag{5.18}$$

is invariant under the evolution operator $S(t)$, where $\rho(C_1)$ was defined in (4.2).

Proof. Let Φ_0 and K_R be as in Lemma 5.5. Let T be as in Lemma 5.5. We prove that there exist two constants $C_1, C_2 > 0$ that do not depend on the truncation parameter R such that the solution Φ obtained in Theorem 5.3 is such that for every time $t \in [0, T]$

(1) $\int_{\mathbb{R}_*} x\Phi(t, \mathrm{d}x) = 1$;
(2) $\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) \leq \max\{C_1, \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi_0(\mathrm{d}x)\}$;
(3) $\int_{\mathbb{R}_*} x^{2-\gamma-\lambda}\Phi(t, \mathrm{d}x) \leq \max\{C_2, \int_{\mathbb{R}_*} x^{2-\gamma-\lambda}\Phi_0(\mathrm{d}x)\}$.

Additionally, we prove that we can conclude from (2) that $\Phi(t, (0, \rho(C_1)) = 0$, for every $t \in [0, T]$.

We first notice that, since $\Phi(t, \cdot)$ has compact support, we can consider in Eq. (5.4) test functions φ such that $\varphi(x) = x^k$, for $k \in \mathbb{R}$.

Let us prove (1). We consider in Eq. (5.4) a test function φ such that $\varphi(x) = x$. We deduce that

$$\frac{d}{dt}\int_0^\infty \xi\Phi(t, \mathrm{d}\xi) = 1 - \int_0^\infty \xi\Phi(t, \mathrm{d}\xi).$$

Hence

$$\int_0^\infty \xi\Phi(t, \xi)d\xi = 1 + \left(\int_{\mathbb{R}_*} \xi\Phi_0(\mathrm{d}\xi) - 1\right)e^{-t}$$

which since $\int_{\mathbb{R}_*} \xi\Phi_0(\mathrm{d}\xi) = 1$ implies

$$\int_{\mathbb{R}_*} \xi\Phi(t, \mathrm{d}\xi) = 1. \tag{5.19}$$

We prove (2). We start by proving this for $\gamma + 2\lambda > 1$. Consider a test function φ such that $\varphi(x) = x^{\gamma+\lambda}$ in Eq. (5.4). Then, we can see that the moment $M_{\gamma+\lambda}$ satisfies the following

$$\begin{aligned}\partial_t M_{\gamma+\lambda} &\leq -\frac{2\lambda+\gamma-1}{1-\gamma}M_{\gamma+\lambda} + \frac{(\gamma+\lambda)}{M_{\gamma+\lambda}}M_{2(\gamma+\lambda)-1}\\ &+ c_3\int_{[\rho(2C_1), \frac{R}{4}]}\int_{[\rho(2C_1), \frac{R}{4}]}(x^{\gamma+\lambda}y^{-\lambda} + x^{-\lambda}y^{\gamma+\lambda})\\ &\times[(x+y)^{\gamma+\lambda} - x^{\gamma+\lambda} - y^{\gamma+\lambda}]\Phi(t, dx)\Phi(t, dy)\end{aligned}$$

since, due to the definition of K_R in (5.3), we have that on the set $[\rho(2C_1), \frac{R}{4}]^2$, there exists a constant $C > 0$ such that $K_R \geq \frac{K}{C}$.

We denote by $z := \frac{y}{x}$. Notice that since $\gamma + \lambda > 0$ when $y \leq x$ we have that

$$\begin{aligned}&\left[x^{\gamma+\lambda}y^{-\lambda} + x^{-\lambda}y^{\gamma+\lambda}\right]\left[(x+y)^{\gamma+\lambda} - x^{\gamma+\lambda} - y^{\gamma+\lambda}\right]\\ &\leq 2x^{2(\gamma+\lambda)}y^{-\lambda}[(1+z)^{\gamma+\lambda} - 1 - z^{\gamma+\lambda}]\\ &\leq 2x^{2(\gamma+\lambda)}y^{-\lambda}\left((\gamma+\lambda)z - z^{\gamma+\lambda}\right)\\ &\leq 2(\gamma+\lambda-1)y^{\gamma}x^{\gamma+\lambda}.\end{aligned}$$

As a consequence, by symmetry, we deduce that

$$\begin{aligned}
\partial_t M_{\gamma+\lambda} &\lesssim -\frac{2\lambda+\gamma-1}{1-\gamma} M_{\gamma+\lambda} + \frac{(\gamma+\lambda)}{M_{\gamma+\lambda}} M_{2(\gamma+\lambda)-1} - 2c_3\,(1-\gamma-\lambda)\, M_\gamma M_{\gamma+\lambda} \\
&\quad + 2c_3(1-\gamma-\lambda)\int_{(\frac{R}{4},\infty)} x^\gamma\,\Phi(\mathrm{d}x) M_{\gamma+\lambda} + 2c_3(1-\gamma-\lambda)\int_{(\frac{R}{4},\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x) M_\gamma \\
&\le -\frac{2\lambda+\gamma-1}{1-\gamma} M_{\gamma+\lambda} + \frac{(\gamma+\lambda)}{M_{\gamma+\lambda}} M_{2(\gamma+\lambda)-1} - 2c_3\,(1-\gamma-\lambda)\, M_\gamma M_{\gamma+\lambda} \\
&\quad + 4^{1-\gamma} R^{\gamma-1} 2c_3(1-\gamma-\lambda) M_{\gamma+\lambda} + c_4 M_\gamma \\
&= -\left[\frac{2\lambda+\gamma-1}{1-\gamma} - \tilde{\varepsilon}\right] M_{\gamma+\lambda} + \frac{1}{M_{\gamma+\lambda}}[(\gamma+\lambda) M_{2(\gamma+\lambda)-1} - 2c_3\,(1-\gamma-\lambda)\, M_\gamma M^2_{\gamma+\lambda} \\
&\quad + c_4 M_{\gamma+\lambda} M_\gamma],
\end{aligned} \tag{5.20}$$

with $\tilde{\varepsilon} := 4^{1-\gamma} R^{\gamma-1} 2c_3(1-\gamma-\lambda)$. Notice we can take $\tilde{R}$ sufficiently large so that $-\frac{2\lambda+\gamma-1}{1-\gamma} + \tilde{\varepsilon} < 0$, for every $R \ge \tilde{R}$.

Since $\gamma < 2(\gamma+\lambda)-1 \le 1$ we deduce, by interpolation, using that $M_1(\Phi(t)) = 1$, for all $t \in [0,T]$, that there exists two positive constants c_5 and c_6 such that

$$M_{2(\gamma+\lambda)-1} \le c_5 M_\gamma + c_6.$$

Hence, since $\gamma + \lambda > 0$ then

$$\begin{aligned}
\partial_t M_{\gamma+\lambda} \le -\left[\frac{2\lambda+\gamma-1}{1-\gamma} - \tilde{\varepsilon}\right] M_{\gamma+\lambda} + \frac{1}{M_{\gamma+\lambda}}[(\gamma+\lambda)\left[c_5 M_\gamma + c_6\right] \\
- 2c_3\,(1-\gamma-\lambda)\, M_\gamma M^2_{\gamma+\lambda} + c_4 M_{\gamma+\lambda} M_\gamma].
\end{aligned}$$

Multiplying by $M_{\gamma+\lambda}$ the inequality we deduce that

$$\begin{aligned}
M_{\gamma+\lambda}\partial_t M_{\gamma+\lambda} \le -\left[\frac{2\lambda+\gamma-1}{1-\gamma} - \tilde{\varepsilon}\right] M^2_{\gamma+\lambda} + [(\gamma+\lambda)\left[c_5 M_\gamma + c_6\right] \\
- 2c_3\,(1-\gamma-\lambda)\, M_\gamma M^2_{\gamma+\lambda} + c_4 M_{\gamma+\lambda} M_\gamma]
\end{aligned}$$

which readjusting the constants implies

$$\partial_t M^2_{\gamma+\lambda} \le -c_3 M^2_{\gamma+\lambda} + M_\gamma \left(c_4 + c_5 M_{\gamma+\lambda} - c_6 M^2_{\gamma+\lambda}\right) + c_7 \tag{5.21}$$

for $c_3, c_4, c_5, c_6, c_7 > 0$. This implies that the set $\left\{\Phi \in \mathcal{M}_+(\mathbb{R}_*) : M_{\gamma+\lambda} \le \frac{c_5 + \sqrt{c_5^2 + 4c_4c_6}}{2c_6}\right\}$ is invariant when $\gamma + 2\lambda > 1$.

We now consider the case $\gamma + 2\lambda = 1$. First of all, notice that, for every $M < R$, we have

$$\int_{[M,\infty)} x^{\gamma+\lambda}\Phi(t,\mathrm{d}x) \le M^{\gamma+\lambda-1}\int_{[M,\infty)} x\Phi(t,\mathrm{d}x) \le M^{\gamma+\lambda-1}.$$

Notice that an upper bound is obvious for the points $t \in [0,T]$ for which $M_{\gamma+\lambda}(\Phi(t)) \le 1$. If there exist $t_1, t_2 \in [0,T]$, $t_1 < t_2$, such that $M_{\gamma+\lambda}(\Phi(t_1)) < 1$ and $M_{\gamma+\lambda}(\Phi(t_2)) > 1$, by the continuity in time of Φ, we have that there exists

$\bar{t} \in [t_1, t_2]$ such that $M_{\gamma+\lambda}(\Phi(\bar{t})) = 1$ and $\varepsilon_1 \in (0, 1)$ such that $M_{\gamma+\lambda}(\Phi(s)) \geq 1$ on $[\bar{t}, \bar{t} + \varepsilon_1]$.

On the interval $[\bar{t}, \bar{t} + \varepsilon_1]$, we can apply the following logic.

We select R large enough so that we can select M to be such that $(1-\delta)^{\frac{1}{1-\gamma-\lambda}} < M < R$, but independent on R, and we deduce that there exists a $\delta > 0$ such that

$$\begin{aligned}\int_{(0,M)} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) &= \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) - \int_{[M,\infty)} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x)\\ &\geq \int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) - M^{\gamma+\lambda-1} \geq 1 - M^{\gamma+\lambda-1} \geq \delta.\end{aligned}$$

As a consequence we deduce that

$$M_\gamma \geq \int_{(0,M)} x^{\gamma}\Phi(t, \mathrm{d}x) \geq M^{-\lambda}\int_{(0,M)} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) \geq \delta M^{-\lambda}. \tag{5.22}$$

Substituting the test function $\varphi(x) = x^{\gamma+\lambda}$ in Eq. (5.4), we deduce that there exists a constant $c > 0$ such that

$$\partial_t M_{\gamma+\lambda} \leq \frac{\gamma+\lambda}{M_{\gamma+\lambda}} M_{2(\gamma+\lambda)-1} - c(1-\gamma-\lambda)M_{\gamma+\lambda}M_\gamma + \tilde{\epsilon}M_{\gamma+\lambda} + cM_\gamma,$$

for $\tilde{\varepsilon} \in (0, 1)$, which can be made sufficiently small as before. Hence, since for suitable constants $c_3, c_4 > 0$ we have that $M_{2(\gamma+\lambda)-1} \leq c_3 M_\gamma + c_4$, similarly to (5.20), we deduce that

$$\frac{1}{2}\partial_t M^2_{\gamma+\lambda} \leq c_5 M_\gamma(\gamma+\lambda) + (\gamma+\lambda)c_6 - c_7 M^2_{\gamma+\lambda}M_\gamma + \tilde{\varepsilon}M^2_{\gamma+\lambda} + c_8 M_\gamma M_{\gamma+\lambda}.$$

Using (5.22) we deduce that there exists a constant $c_9 > 0$ such that

$$\frac{1}{2}\partial_t M^2_{\gamma+\lambda} \leq M_\gamma\left[c_5(\gamma+\lambda) + (\gamma+\lambda)c_9 + c_8 M_{\gamma+\lambda} - (c_7 - \tilde{\varepsilon}\delta^{-1}M^\lambda)M^2_{\gamma+\lambda}\right].$$

Choosing now $\tilde{\varepsilon}$ sufficiently small such that $-c_7 + \tilde{\varepsilon}\delta^{-1}M^\lambda < 0$, we have that there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(t, \mathrm{d}x) \leq C_1. \tag{5.23}$$

We now prove that $(S(t)\Phi_0)((0, \rho(C_1))) = 0$. Notice that, by Lemma 5.5, we know that $(S(t)\Phi_0)((0, \rho(2C_1))) = 0$, where we recall that $\rho(2C_1) \leq \rho(C_1)$. Consider now a sequence of test functions $\{\varphi_n\}$ such that, for every $n \in \mathbb{N}$, φ_n is decreasing and supported in $(0, \rho(C_1))$. Since φ_n is decreasing, we deduce that the solution F of Eq. (5.10) satisfies

$$\int_{\mathbb{R}_*} \varphi_n(x)F(t, \mathrm{d}x) \leq \int_{\mathbb{R}_*} \varphi_n(x)\Phi_0(\mathrm{d}x) + C(R)\int_0^t \int_{\mathbb{R}_*} \varphi_n(x)F(s, \mathrm{d}x)\mathrm{d}s,$$
$$\text{for every } n \in \mathbb{N},$$

where we used in addition that $M_1(F)$ is bounded. Using Grönwall's inequality, we deduce that

$$\int_{\mathbb{R}_*} \varphi_n(x) F(t, \mathrm{d}x) = 0,$$

for every $t \in [0, T]$ and $n \in \mathbb{N}$.

We then let φ_n converge pontwise to $\chi_{(0,\rho(C_1))}$ and obtain that

$$F(t, (0, \rho(C_1))) = 0,$$

for every $t \in [0, T]$.

Consider now a test function φ such that $\varphi(x) = 0$ for every $x \geq \rho(C_1)$ in equality (5.14). Then

$$\int_{\mathbb{R}_*} \varphi(x)\Phi(t, \mathrm{d}x) = e^{\frac{1+\gamma}{1-\gamma}t} \int_{[\rho(C_1),\infty)} \varphi(X(t, y, \alpha))F(t, \mathrm{d}y).$$

Using the fact that $X(t, y, \alpha) \geq \rho(C_1)$ for every $y \geq \rho(C_1)$, we deduce that $\Phi(t, (0, \rho(C_1))) = 0$, for every $t \in [0, T]$.

We conclude by proving (3). We consider a test function φ equal to $x^{2-\gamma-\lambda}$ in Eq. (5.4) and deduce that

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}_*} \xi^{2-\gamma-\lambda}\Phi(t, \mathrm{d}\xi) - \frac{3-3\gamma-2\lambda}{1-\gamma} \int_{\mathbb{R}_*} \xi^{2-\gamma-\lambda}\Phi(t, \mathrm{d}\xi) - \frac{2-\gamma-\lambda}{M_{\gamma+\lambda}(\Phi(t))} M_1(\Phi(t)) \\
&= \frac{1}{2} \int_{[\rho(C_1),\infty)} \int_{[\rho(C_1),\infty)} K_R(\xi, z) \left[(z+\xi)^{2-\gamma-\lambda} - \xi^{2-\gamma-\lambda} - z^{2-\gamma-\lambda}\right] \Phi(t, \mathrm{d}\xi)\Phi(t, \mathrm{d}z).
\end{aligned} \tag{5.24}$$

By Cauchy–Schwarz inequality it follows that

$$\frac{1}{\int_{[\rho(C_1),\infty)} \xi^{\gamma+\lambda}\Phi(t, \mathrm{d}\xi)} \leq \frac{\int_{[\rho(C_1),\infty)} \xi^{2-\gamma-\lambda}\Phi(t, \mathrm{d}\xi)}{\left(\int_{[\rho(C_1),\infty)} \xi\Phi(t, \mathrm{d}\xi)\right)^2} = \int_{[\rho(C_1),\infty)} \xi^{2-\gamma-\lambda}\Phi(t, \mathrm{d}\xi). \tag{5.25}$$

Plugging (5.25) into Eq. (5.24) and using the fact that the total mass is equal to 1 proved in (5.19), we deduce that

$$\begin{aligned}
&\partial_t \int_{[\rho(C_1),\infty)} \xi^{2-\gamma-\lambda}\Phi(t, \mathrm{d}\xi) \leq \left(\frac{3\gamma-3+2\lambda}{1-\gamma} + 2 - \gamma - \lambda\right) \int_{[\rho(C_1),\infty)} \xi^{2-\gamma-\lambda}\Phi(t, \mathrm{d}\xi) \\
&+ \frac{1}{2} \int_{[\rho(C_1),\infty)} \int_{[\rho(C_1),\infty)} K_R(\xi, z) \left[(z+\xi)^{2-\gamma-\lambda} - \xi^{2-\gamma-\lambda} - z^{2-\gamma-\lambda}\right] \Phi(t, \mathrm{d}\xi)\Phi(t, \mathrm{d}z).
\end{aligned}$$

Hence, since $-1 < \gamma < 1$ and $0 < \gamma + \lambda < 1$, we have

$$\frac{3\gamma-3+2\lambda}{1-\gamma} + 2 - \gamma - \lambda = -\frac{(\gamma+1)(1-\gamma-\lambda)}{1-\gamma} < 0.$$

By symmetry,

$$\frac{1}{2}\int_{[\rho(C_1),\infty)}\int_{[\rho(C_1),\infty)} K_R(\xi,z)\left[(z+\xi)^{2-\gamma-\lambda}-\xi^{2-\gamma-\lambda}-z^{2-\gamma-\lambda}\right]\Phi(t,d\xi)\\ \times\Phi(t,dz)\\ \leq \int_{[\rho(C_1),\infty)}\int_{[\rho(C_1),z]} K_R(\xi,z)\left[(z+\xi)^{2-\gamma-\lambda}-\xi^{2-\gamma-\lambda}-z^{2-\gamma-\lambda}\right]\\ \times\Phi(t,d\xi)\Phi(t,dz).$$

Assume $\xi\leq z$. Denote $\eta:=\frac{\xi}{z}\in(0,1]$ and observe

$$\begin{aligned}K_R(\xi,z)&[(\xi+z)^{2-\gamma-\lambda}-\xi^{2-\gamma-\lambda}-z^{2-\gamma-\lambda}]\\ &\leq K(\xi,z)z^{2-\gamma-\lambda}[(1+\eta)^{2-\gamma-\lambda}-1-\eta^{2-\gamma-\lambda}]\\ &\leq CK(\xi,z)z^{2-\gamma-\lambda}\eta\\ &\leq Cz^{\gamma}(\eta^{\gamma+\lambda}+\eta^{-\lambda})z^{2-\gamma-\lambda}\eta\leq 2Cz^{2-\lambda}\eta^{1-\lambda}\\ &\leq 4C(z\xi^{1-\lambda}+\xi z^{1-\lambda}).\end{aligned}\tag{5.26}$$

Since $\rho(C_1)\leq\xi$ and $\rho(C_1)\leq z$, then $z^{1-\lambda}\leq\rho(C_1)^{-\lambda}z$. Hence

$$\frac{d}{dt}\int_{\mathbb{R}_*}\xi^{2-\gamma-\lambda}\Phi(t,\mathrm{d}\xi)\leq -c_3\int_{\mathbb{R}_*}\xi^{2-\gamma-\lambda}\Phi(t,\mathrm{d}\xi)+c(\rho(C_1)),$$

for suitable constants $c_3, c(\rho(C_1))>0$. Then, (3) follows. □

Proposition 5.8. (Time-continuity of the semigroup) *Assume K_R to be a truncated kernel as in* (5.3) *defined as a function of the homogeneous symmetric kernel K that satisfies* (1.5), (1.6), *for parameters $\gamma,\lambda\in\mathbb{R}$ such that* (1.7) *holds and such that $\gamma>-1$ and $\gamma+2\lambda\geq 1$. Let $\Phi_0\in P$. Let $T>0$ be as in Theorem* 5.3. *The map $S(\cdot)\Phi_0:[0,T]\to P$ is continuous in time, for every fixed Φ_0, where P was defined in* (5.18).

Proof. Let $T>0$. We want to estimate the value of $|S(t)\Phi_0-S(s)\Phi_0|$, for $s,t\in[0,T]$. Assume without loss of generality that $s\leq t$. By the definition of the operator S we know that

$$\int_{\mathbb{R}_*}\varphi(x)[\Phi(t,\mathrm{d}x)-\Phi(s,\mathrm{d}x)]-\frac{1+\gamma}{1-\gamma}\int_s^t\int_{\mathbb{R}_*}\varphi(x)\Phi(z,\mathrm{d}x)\mathrm{d}z\\ +\frac{2}{1-\gamma}\int_s^t\int_{\mathbb{R}_*}\varphi'(x)x\Phi(z,\mathrm{d}x)\mathrm{d}z\\ -\int_s^t\frac{1}{\int_{\mathbb{R}_*}x^{\gamma+\lambda}\Phi(z,\mathrm{d}x)}\int_{\mathbb{R}_*}\varphi'(x)x^{\gamma+\lambda}\Phi(z,\mathrm{d}x)\mathrm{d}z\\ =\frac{1}{2}\int_s^t\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}K_R(x,y)\left[\varphi(x+y)-\varphi(x)-\varphi(y)\right]\Phi(z,\mathrm{d}x)\Phi(z,\mathrm{d}y)\mathrm{d}z.$$

We have that there exists a constant $C_2 > 0$ such that $M_{\gamma+\lambda}(\Phi(r)) \geq \frac{1}{C_2}$, for every $r \in [0, T]$.

As Φ has compact support, we can conclude that the exists a constant $C > 0$, which may depend on $\rho(C_1)$, R, T, γ, λ, but independent on s, t, such that

$$\left| \int_{\mathbb{R}_*} \varphi(x)[\Phi(t, \mathrm{d}x) - \Phi(s, \mathrm{d}x)] \right| \leq C|t-s|,$$

thus giving us the desired continuity of the semigroup. □

Lemma 5.9. (Dual equation) *Assume K_R to be a truncated kernel as in (5.3) defined as a function of a homogeneous symmetric kernel K that satisfies (1.5), (1.6), for parameters $\gamma, \lambda \in \mathbb{R}$ satisfying (1.7)0 and with $\gamma > -1$ and $\gamma + 2\lambda \geq 1$. Let $\Phi_1, \Phi_2 \in P$ be two solutions of (5.4) with initial conditions $\Phi_{\text{in},1}, \Phi_{\text{in},2} \in P$, respectively, and $T > 0$ be as in Theorem 5.3. Then there exists a unique solution $\varphi \in \mathrm{C}^1([0, T], \mathrm{C}_c^1([\rho(C_1), \infty))$, with $\varphi(T, \cdot) = \psi(\cdot)$, where ψ is an arbitrary function in $\mathrm{C}_c^1([\rho(C_1), \infty))$, which solves the following equation:*

$$\begin{aligned} &\partial_t \varphi(t, \xi) + \frac{1+\gamma}{1-\gamma} \varphi(t, \xi) - \frac{2\xi}{1-\gamma} \partial_\xi \varphi(t, \xi) + \frac{\xi^{\gamma+\lambda}}{M_{\gamma+\lambda}(\Phi_1)} \partial_\xi \varphi(t, \xi) \\ &+ \mathbb{L}(\varphi)(t, \xi) = 0, \end{aligned} \tag{5.27}$$

where

$$\begin{aligned} \mathbb{L}(\varphi)(t, \xi) :=& - \frac{\xi^{\gamma+\lambda}}{M_{\gamma+\lambda}(\Phi_1) M_{\gamma+\lambda}(\Phi_2)} \int_{\mathbb{R}_*} \partial_z \varphi(t, z) z^{\gamma+\lambda} \Phi_2(t, \mathrm{d}z) \\ &+ \frac{1}{2} \int_{\mathbb{R}_*} K_R(\xi, \eta)[\varphi(t, \xi+\eta) - \varphi(t, \xi) - \varphi(t, \eta)](\Phi_1(t, \mathrm{d}\eta) \\ &+ \Phi_2(t, \mathrm{d}\eta)). \end{aligned}$$

Remark 5.10. We prove the statement of the lemma for a modified operator $\overline{\mathbb{L}}(\varphi)$ which preserves the continuity in the variable ξ, is equal to zero when $\xi \geq 8R$ and $\overline{\mathbb{L}}(\varphi) = \mathbb{L}(\varphi)$ if $\xi \in [\rho(C_1), 4R]$. Due to the support of Φ_1 and Φ_2, when proving the continuity of the map $S(t)$ in the weak-* topology, it suffices to analyse the operator $\overline{\mathbb{L}}(\varphi)$. Thus, it is enough to prove the statement of the lemma only for the operator $\overline{\mathbb{L}}(\varphi)$. Notice that, for example, the operator $\mathbb{L}(\varphi)$ does not preserve compactness because of the presence of the term $\frac{\xi^{\gamma+\lambda}}{M_{\gamma+\lambda}(\Phi_1) M_{\gamma+\lambda}(\Phi_2)} \int_{\mathbb{R}_*} \partial_z \varphi(t, z) z^{\gamma+\lambda} \Phi_2(t, \mathrm{d}z)$. We keep the notation $\mathbb{L}(\varphi)$ for simplicity. We omit further details as the proof consists of standard methods used in the study of coagulation equations, see, for example, [10].

Proof of Lemma 5.9. First, we use the method of characteristics. We define $X(t, \xi)$ to be the solution of the ODE

$$x'(t) = -\frac{2}{1-\gamma} x + \frac{1}{\int_{\mathbb{R}_*} z^{\gamma+\lambda} \Phi_1(t, \mathrm{d}z)} x^{\gamma+\lambda}$$

with initial condition $x(0) = \xi$.

In this way, Eq. (5.27) can be rewritten in the fixed point form

$$\varphi(t, X(t,\xi)) = \mathcal{L}[\varphi](t,\xi), \tag{5.28}$$

where

$$\mathcal{L}[\varphi](t,\xi) := \varphi(T, X(T,\xi)) + \frac{1+\gamma}{1-\gamma}\int_t^T \varphi(s, X(s,\xi))\mathrm{d}s + \int_t^T \mathbb{L}(\varphi)(s, X(s,\xi))\mathrm{d}s.$$

Our strategy for proving the statement of the lemma is to apply Banach fixed point theorem. The operator $\mathcal{L}$ maps $C^1([0,T], \mathrm{C}_c^1([\rho(C_1),\infty)))$ in itself.

We prove that the operator $\mathcal{L}$ is a contraction if we endow $Y := C([0,T], C_c^1([\rho(C_1),\infty))$ with the norm $\|\varphi\|_Y := \sup_{t\in[0,T]}\big(\sup_{x\in\mathbb{R}_*}|\varphi(t,x)| + \sup_{x\in\mathbb{R}_*}|\partial_x\varphi(t,x)|\big)$.

To this end, we notice that

$$\mathcal{L}[\varphi_1](t,\xi) - \mathcal{L}[\varphi_2](t,\xi) = \frac{1+\gamma}{1-\gamma}\int_t^T (\varphi_1(s,X(s,\xi)) - \varphi_2(s,X(s,\xi)))\,\mathrm{d}s + \int_t^T [\mathbb{L}(\varphi_1)(s,X(s,\xi)) - \mathbb{L}(\varphi_2)(s,X(s,\xi))]\,\mathrm{d}s.$$

We notice that

$$\begin{aligned}
&\int_t^T [\mathbb{L}(\varphi_1)(s,X(s,\xi)) - \mathbb{L}(\varphi_2)(s,X(s,\xi))]\,\mathrm{d}s \\
&= -\int_t^T \frac{1}{M_{\gamma+\lambda}(\Phi_1)M_{\gamma+\lambda}(\Phi_2)}\int_{\mathbb{R}_*}(\partial_z\varphi_1(s,z) - \partial_z\varphi_2(s,z))\,z^{\gamma+\lambda}\Phi_2(s,\mathrm{d}z)X(s,\xi)^{\gamma+\lambda}\mathrm{d}s \\
&\quad + \frac{1}{2}\int_t^T\int_{\mathbb{R}_*} K_R(X(s,\xi),\eta)[\varphi_1(s,X(s,\xi)+\eta) - \varphi_2(s,X(s,\xi)+\eta) - \varphi_1(s,X(s,\xi)) \\
&\quad + \varphi_2(s,X(s,\xi)) - \varphi_1(s,\eta) + \varphi_2(s,\eta)](\Phi_1(s,\mathrm{d}\eta) + \Phi_2(s,\mathrm{d}\eta))\mathrm{d}s.
\end{aligned}$$

From this we deduce that

$$\|\mathcal{L}[\varphi_1] - \mathcal{L}[\varphi_2]\|_Y \le Tc(\rho(C_1), R, \Phi_1, \Phi_2)\|\varphi_1 - \varphi_2\|_Y$$

and hence $\mathcal{L}$ is a contraction for sufficiently small times T. We can extend the solution to all possible times noting that the contraction constant $c(\rho(C_1), \Phi_1, \Phi_2, R)$ does not depend on the final condition ψ. We thus deduce that there exists a solution φ of the fixed point $\varphi = \mathcal{L}[\varphi]$. □

We now prove that the found solution is Lipschitz continuous.

Proposition 5.11. *Assume K_R to be a truncated kernel as in (5.3) defined as a function of a homogeneous symmetric kernel K such that it satisfies (1.5), (1.6), for parameters $\gamma, \lambda \in \mathbb{R}$ satisfying (1.7) and such that $\gamma > -1$ and $\gamma + 2\lambda \ge 1$. Let $T > 0$ be as in Theorem 5.3. Let $\varphi \in \mathrm{C}^1([0,T], \mathrm{C}^1([\rho(C_1), 8R]))$ with initial datum $\varphi(T,\cdot)$ be the function found in Lemma 5.9. Assume, in addition, that*

$\sup_{\xi\in[\rho(C_1),8R]}|\varphi(T,\xi)|\le 1$ *and that* $\varphi(T,\xi)$ *is Lipschitz. Then* φ *is Lipschitz continuous, in the sense that, for every* $t\in[0,T]$, *there exists* $C(t)>0$ *such that*

$$\sup_{s\in[0,t]}|\varphi(s,\xi)-\varphi(s,\tilde{\xi})|\le C(t)|\xi-\tilde{\xi}|,$$

for every $\xi,\tilde{\xi}\in[\rho(C_1),8R]$. *Moreover,* $C(t)$ *may depend on the norm of* Φ_1 *and* Φ_2*, but is otherwise independent of the choice of* Φ_1 *and* Φ_2.

Proof. Notice first that, since $\sup_{\xi\in[\rho(C_1),8R]}|\varphi(T,\xi)|\le 1$, there exists a constant $C>0$, which depends on the norm of Φ_1 and Φ_2 and the parameters $\rho(C_1)$, R and t, such that $\sup_{s\in[0,t],\xi\in[\rho(C_1),8R]}|\varphi(s,\xi)|\le C$. This can be seen by looking at (5.28). We will use Grönwall's inequality in (5.28) in order to prove that φ is Lipschitz:

$$\begin{aligned}|\varphi(t,X(t,\xi))-\varphi(t,X(t,\tilde{\xi}))|&\le|\varphi(T,X(T,\xi))-\varphi(T,X(T,\tilde{\xi}))|\\&\quad+\frac{1+\gamma}{1-\gamma}\int_t^T|\varphi(s,X(s,\xi))-\varphi(s,X(s,\tilde{\xi}))|\mathrm{d}s\\&\quad+\int_t^T|\mathbb{L}[\varphi](s,X(s,\xi))-\mathbb{L}[\varphi](s,X(s,\tilde{\xi}))|\mathrm{d}s.\end{aligned}$$

In order to bound the term

$$\int_t^T|\mathbb{L}[\varphi](s,X(s,\xi))-\mathbb{L}[\varphi](s,X(s,\tilde{\xi}))|\mathrm{d}s,$$

we need to estimate

$$\begin{aligned}I_1=\Bigg|&\frac{1}{2}\int_{\mathbb{R}_*}K_R(X(t,\xi),\eta)[\varphi(t,X(t,\xi)+\eta)\\&-\varphi(t,X(t,\xi))-\varphi(t,\eta)](\Phi_1(t,\mathrm{d}\eta)+\Phi_2(t,\mathrm{d}\eta))\\&-\frac{1}{2}\int_{\mathbb{R}_*}K_R(X(t,\tilde{\xi}),\eta)[\varphi(t,X(t,\tilde{\xi})+\eta)\\&-\varphi(t,X(t,\tilde{\xi}))-\varphi(t,\eta)](\Phi_1(t,\mathrm{d}\eta)+\Phi_2(t,\mathrm{d}\eta))\Bigg|\end{aligned}$$

and

$$I_2=\frac{|X(t,\xi)^{\gamma+\lambda}-X(t,\tilde{\xi})^{\gamma+\lambda}|}{M_{\gamma+\lambda}(\Phi_1)M_{\gamma+\lambda}(\Phi_2)}\int_{\mathbb{R}_*}|\partial_z\varphi(t,z)|z^{\gamma+\lambda}\Phi_2(t,\mathrm{d}z).$$

For I_1, by the definition of K_R in (5.3), we have K_R is C^1 and we can assume it has compact support as we are only interested in the region $[\rho(C_1),8R]^2$. Thus, we have that the first derivative of K_R is bounded from above. Moreover, there exist constants $L_1(t),L_2(t)>0$ such that $L_1(t)|\xi-\tilde{\xi}|\le|X(t,\xi)-X(t,\tilde{\xi})|\le L_2(t)|\xi-\tilde{\xi}|$.

For I_2, we use

$$\frac{1}{M_{\gamma+\lambda}(\Phi_1)M_{\gamma+\lambda}(\Phi_2)}\int_{\mathbb{R}_*}|\partial_z\varphi(t,z)|z^{\gamma+\lambda}\Phi_2(t,dz)$$
$$\leq \frac{C_1}{(C_2)^2}\sup_{z\in[\rho(C_1),8R]}|\partial_z\varphi(t,z)|.$$

If we prove that there exists a constant $C>0$, which can depend on $\rho(C_1)$, R, and the norms of Φ_1, and Φ_2, but does not vary depending on the choice of Φ_1,Φ_2 such that

$$\sup_{z\in[\rho(C_1),8R]}|\partial_z\varphi(t,z)|\leq C, \tag{5.29}$$

then there exists a constant $C>0$, which can depend on $\rho(C_1)$, R and the norms of Φ_1 and Φ_2 such that

$$|\varphi(t,X(t,\xi))-\varphi(t,X(t,\tilde{\xi}))|$$
$$\leq C|X(t,\xi)-X(t,\tilde{\xi})|+C\int_t^T|\varphi(s,X(s,\xi))-\varphi(s,X(s,\tilde{\xi}))|. \tag{5.30}$$

Let us now prove (5.29). From Eq. (5.28) and the fact that K_R is in C^1 with compact support, we obtain an upper bound for $|\partial_2\varphi(t,X(t,\xi))|$.

We then use that there exist some constants $\overline{c}(t),\overline{\overline{c}}(t)>0$ such that $\overline{c}(t)\xi\leq X(t,\xi)\leq\overline{\overline{c}}(t)\xi$, with $\xi\in[\rho(C_1),8R]$ in order to obtain the desired bound for $\sup_{z\in[\rho(C_1),8R]}|\partial_z\varphi(t,z)|$.

We use Grönwall's inequality in (5.30) and obtain that $|\varphi(t,X(t,\xi))-\varphi(t,X(t,\tilde{\xi}))|\leq C(t)|X(t,\xi)-X(t,\tilde{\xi})|$. Thus, we can conclude that also $|\varphi(t,\xi)-\varphi(t,\tilde{\xi})|\leq C(t)|\xi-\tilde{\xi}|$ since $X(t,\xi)$ is Lipschitz continuous in the ξ variable. □

Proposition 5.12. (Continuity of the semigroup in the weak topology) *Assume K_R to be a truncated kernel as in (5.3) defined as a function of a homogeneous symmetric kernel K such that it satisfies (1.5), (1.6), for parameters $\gamma,\lambda\in\mathbb{R}$ satisfying (1.7) and such that $\gamma>-1$ and $\gamma+2\lambda\geq 1$. For every time $t>0$ the map*

$$S(t):P\rightarrow P$$

is continuous in the weak- topology, where P was defined in (5.18).*

Proof. Let $\delta>0$. In order to prove continuity in the weak-* topology of the semigroup and because of the support of Φ_1 and Φ_2, it is enough to prove that, if for every $\psi\in C_c([\rho(C_1),\infty))$, with $||\psi||_\infty\leq 1$, we have that

$$\int_{\mathbb{R}_*}\psi(x)\left(\Phi_{\text{in},1}(\mathrm{d}x)-\Phi_{\text{in},2}(\mathrm{d}x)\right)\text{ is sufficiently small,} \tag{5.31}$$

then we have that for every $\psi \in C_c([\rho(C_1), \infty))$, with $||\psi||_\infty \leq 1$,

$$\int_{\mathbb{R}_*} \psi(x)\,(\Phi_1(t, \mathrm{d}x) - \Phi_2(t, \mathrm{d}x)) \leq \delta,$$

where Φ_1 and Φ_2 are the solutions of Eq. (5.4) with initial conditions $\Phi_{\mathrm{in},1}$ and $\Phi_{\mathrm{in},2}$, respectively. To this end we notice that since Φ_1 and Φ_2 satisfy Eq. (5.4) then

$$\begin{aligned}
&\int_{\mathbb{R}_*} \varphi(t,x)[\Phi_1(t,\mathrm{d}x) - \Phi_2(t,\mathrm{d}x)] - \int_{\mathbb{R}_*} \varphi(0,x)[\Phi_{\mathrm{in},1}(\mathrm{d}x) - \Phi_{\mathrm{in},2}(\mathrm{d}x)] \\
&\quad - \int_0^t \int_{\mathbb{R}_*} \partial_s \varphi(s,x)[\Phi_1(s,\mathrm{d}x) - \Phi_2(s,\mathrm{d}x)]\mathrm{d}s \\
&\quad - \frac{1+\gamma}{1-\gamma} \int_0^t \int_{\mathbb{R}_*} \varphi(s,x)[\Phi_1(s,\mathrm{d}x) - \Phi_2(s,\mathrm{d}x)]\mathrm{d}s \\
&\quad + \frac{2}{1-\gamma} \int_0^t \int_{\mathbb{R}_*} \partial_x \varphi(s,x) x [\Phi_1(s,\mathrm{d}x) - \Phi_2(s,\mathrm{d}x)]\mathrm{d}s \\
&\quad - \int_0^t \frac{1}{\int_{\mathbb{R}_*} z^{\gamma+\lambda}\Phi_1(s,\mathrm{d}z)} \int_{\mathbb{R}_*} \partial_x \varphi(s,x) x^{\gamma+\lambda} \Phi_1(s,\mathrm{d}x)\mathrm{d}s \\
&\quad + \int_0^t \frac{1}{\int_{\mathbb{R}_*} z^{\gamma+\lambda}\Phi_2(s,\mathrm{d}z)} \int_{\mathbb{R}_*} \partial_x \varphi(s,x) x^{\gamma+\lambda} \Phi_2(s,\mathrm{d}x)\mathrm{d}s \\
&= \frac{1}{2} \int_0^t \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K_R(x,y) \chi_\varphi(s,x,y)[\Phi_1(s,\mathrm{d}y) \\
&\quad + \Phi_2(s,\mathrm{d}y)][\Phi_1(s,\mathrm{d}x) - \Phi_2(s,\mathrm{d}x)]\mathrm{d}s,
\end{aligned}$$

where $\chi_\varphi(s,x,y) := \varphi(s,x+y) - \varphi(s,x) - \varphi(s,y)$, for every $\varphi \in C([0,t], C_c^1(\mathbb{R}_*))$. To simplify the computations, we adopt the notation $\tilde{\Phi} := \Phi_1 - \Phi_2$ and we notice that $\tilde{\Phi}$ satisfies the following equation for every $\varphi \in C^1([0,t], C_c^1(\mathbb{R}_*))$

$$\begin{aligned}
&\int_{\mathbb{R}_*} \varphi(t,x)\tilde{\Phi}(t,\mathrm{d}x) - \int_{\mathbb{R}_*} \varphi(0,x)[\Phi_{\mathrm{in},1}(\mathrm{d}x) - \Phi_{\mathrm{in},2}(\mathrm{d}x)] \\
&\quad - \int_0^t \int_{\mathbb{R}_*} \partial_s \varphi(s,x)\tilde{\Phi}(s,\mathrm{d}x)\mathrm{d}s - \frac{1+\gamma}{1-\gamma} \int_0^t \int_{\mathbb{R}_*} \varphi(s,x)\tilde{\Phi}(s,\mathrm{d}x)\mathrm{d}s \\
&\quad + \frac{2}{1-\gamma} \int_0^t \int_{\mathbb{R}_*} \partial_x \varphi(s,x) x \tilde{\Phi}(s,\mathrm{d}x)\mathrm{d}s \\
&\quad - \int_0^t \frac{1}{\int_{\mathbb{R}_*} z^{\gamma+\lambda}\Phi_1(s,\mathrm{d}z)} \int_{\mathbb{R}_*} \partial_x \varphi(s,x) x^{\gamma+\lambda} \tilde{\Phi}(s,\mathrm{d}x)\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{R}_*} \mathbb{L}[\varphi](s,x)\tilde{\Phi}(s,\mathrm{d}x)\mathrm{d}s.
\end{aligned}$$

First, we make the following notation:

$$C_{norm} := \sup_{s\in[0,t]} \int_{\mathbb{R}_*} [\Phi_1 + \Phi_2](s,\mathrm{d}x) < \infty.$$

Notice that C_{norm} can be bounded from above by

$$C_{norm} = \sup_{s\in[0,t]} \int_{\mathbb{R}_*} [\Phi_1 + \Phi_2](s, \mathrm{d}x) \leq C(R, \rho(C_1), t) \int_{\mathbb{R}_*} [\Phi_{\text{in},1} + \Phi_{\text{in},2}](\mathrm{d}x).$$

Let now φ be the solution found in Lemma 5.9 with coagulation kernel K_R. By Proposition 5.11, we know that there exists a constant $C(t, R, \Phi_1, \Phi_2)$, that depends only on the norm of Φ_1 and Φ_2, but is otherwise independent of the choice of Φ_1 and Φ_2, such that $|\varphi(s,\xi) - \varphi(s,\tilde{\xi})| \leq C(t, R, \Phi_1, \Phi_2)|\xi - \tilde{\xi}|$, for every $\xi, \tilde{\xi} \in [\rho(C_1), 8R]$ and $s \in [0, t]$. We thus look at the set:

$$\begin{aligned}\mathcal{K}_{\rho(C_1),R} :=& \{\chi \in \mathrm{C}([\rho(C_1), 8R]) \,|\, |\chi(\xi) - \chi(\tilde{\xi})| \leq C(t, R, \Phi_1, \Phi_2)|\xi - \tilde{\xi}|,\\ & \text{for all } \xi, \tilde{\xi} \in [\rho(C_1), 8R]\} \subset \mathrm{C}([\rho(C_1), 8R]).\end{aligned}$$

The set $\mathcal{K}_{\rho(C_1),R}$ is totally bounded, thus there exist $N \in \mathbb{N}$ and $\chi_1, \ldots, \chi_N \in \mathcal{K}_{\rho(C_1),R}$ such that $\mathcal{K}_{\rho(C_1),R} \subseteq \cup_{i=1}^N B(\chi_i, \frac{\delta}{4NC_{norm}})$.

Then we obtain that

$$\int_{\mathbb{R}_*} \varphi(t,x)\tilde{\Phi}(t, \mathrm{d}x) = \int_{\mathbb{R}_*} \varphi(0,x)[\Phi_{\text{in},1}(\mathrm{d}x) - \Phi_{\text{in},2}(\mathrm{d}x)] =: T_1. \tag{5.32}$$

We can bound T_1 by

$$\begin{aligned} T_1 \leq & \min_{i=1}^{N} \int_{\mathbb{R}_*} |[\varphi(0,x) - \chi_i(x)][\Phi_{\text{in},1}(\mathrm{d}x) - \Phi_{\text{in},2}(\mathrm{d}x)]| \\ & + \max_{i=1}^{N} \Big(\int_{\mathbb{R}_*} \chi_i(x)[\Phi_{\text{in},1}(\mathrm{d}x) - \Phi_{\text{in},2}(\mathrm{d}x)] \Big) \\ \leq & \frac{\delta}{4C_{norm}} \int_{\mathbb{R}_*} [\Phi_{\text{in},1}(\mathrm{d}x) + \Phi_{\text{in},2}(\mathrm{d}x)] + \max_{i=1}^{N} \int_{\mathbb{R}_*} \chi_i(x)[\Phi_{\text{in},1}(\mathrm{d}x) \\ & - \Phi_{\text{in},2}(\mathrm{d}x)] \leq \delta, \end{aligned}$$

where in the last step we used (5.31). □

Proof of Theorem 5.6. By Proposition 5.12 we know that the operator $\Phi \mapsto S(t)\Phi$ is continuous in the weak-* topology. Additionally, thanks to Proposition 5.7, we know that P is an invariant region for $S(t)$. Since P is also convex and compact and since we have proven that the map $t \mapsto S(t)$ is continuous, we apply Theorem 1.2 in [8] to deduce that there exists a Φ such that $S(t)\Phi = \Phi$. □

5.3. Existence of the self-similar profile

To keep the notation lighter, in the previous sections we denoted with Φ the solution of the truncated problem. In this section, since we pass to the limit as R tends to infinity, we add the label R to Φ and we will denote with Φ the self-similar profile. For simplicity, we denote by $\rho := \rho(C_1)$, where $\rho(C_1)$ was defined in (4.2).

Proof of Theorem 5.1. For every $R > 1$, sufficiently large, there exist $T > 0$ and $\Phi_R \in C^1([0,T]; \mathcal{M}_+(\mathbb{R}_*))$ such that

$$\int_{\mathbb{R}_*} \varphi(x)\dot{\Phi}_R(t,\mathrm{d}x) - \frac{1+\gamma}{1-\gamma}\int_{\mathbb{R}_*} \varphi(x)\Phi_R(t,\mathrm{d}x) + \frac{2}{1-\gamma}\int_{\mathbb{R}_*} \varphi'(x)x\Phi_R(t,\mathrm{d}x)$$
$$- \frac{1}{M_{\gamma+\lambda}(\Phi_R(t))}\int_{\mathbb{R}_*} \varphi'(x)x^{\gamma+\lambda}\Phi_R(t,\mathrm{d}x)$$
$$= \frac{1}{2}\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} K_R(x,y)\,[\varphi(x+y)-\varphi(x)-\varphi(y)]\,\Phi_R(t,\mathrm{d}x)\Phi_R(t,\mathrm{d}y),$$

where K_R is the coagulation kernel defined in (5.3). Notice that the bounds in Proposition 5.7 are independent of $R > 1$.

Thanks to Theorem 5.6 we know that a measure $\Phi_R \in \mathcal{M}_+(\mathbb{R}_*)$ satisfying Eq. (5.16) and satisfying the bounds (5.17) exists. We notice also that, since $\gamma+\lambda > 0$,

$$\int_{(0,\infty)} \Phi_R(\mathrm{d}x) = \int_{[\rho,\infty)} \Phi_R(\mathrm{d}x) \le \rho^{-\gamma-\lambda}\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) \le \rho^{-\gamma-\lambda}C_1. \tag{5.33}$$

Hence, Banach-Alaoglu Theorem implies that there exists Φ such that

$$\Phi_R \rightharpoonup \Phi \text{ as } R \to \infty \tag{5.34}$$

in the weak-* topology.

We now prove that the measure Φ in (5.34) satisfies Eq. (3.3) by taking the limit as $R \to \infty$. Fix $\varphi \in C_c(\mathbb{R}_*)$. We start with passing to the limit in the coagulation term.

Let $\varepsilon \in (0,1)$. We first show that

$$\left|\int_{[\rho,\infty)}\int_{[\rho,\infty)} (K(\xi,z)-K_R(\xi,z))\,\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)]\right|$$
$$\le \varepsilon, \tag{5.35}$$

for sufficiently large R. We then prove that

$$\int_{[\rho,\infty)}\int_{[\rho,\infty)} K(\xi,z)\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)] \tag{5.36}$$

converges to

$$\int_{[\rho,\infty)}\int_{[\rho,\infty)} K(\xi,z)\Phi(\mathrm{d}\xi)\Phi(\mathrm{d}z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)] \tag{5.37}$$

as $R \to \infty$.

For (5.35), we have that:

$$\begin{aligned}
&\left|\int_{[\rho,\infty)}\int_{[\rho,\infty)}(K(\xi,z)-K_R(\xi,z))\,\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)]\right| \\
&\quad\le \int_{[\rho,\frac{R}{4}]}\int_{[\rho,\frac{R}{4}]}\left|(K(\xi,z)-K_R(\xi,z))\,\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)]\right| \\
&\qquad+2\int_{(\frac{R}{4},\infty)}\int_{[\rho,\infty)}\left|(K(\xi,z)-K_R(\xi,z))\,\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)]\right| \\
&\quad= T_1+T_2.
\end{aligned}$$

For the first term, we have

$$T_1 \le e^{-R}3||\varphi||_\infty\rho^{-2},$$

where we used the definition of K_R in (5.3) and the fact that the total mass of the measures is equal to 1.

For the region $(\frac{R}{4},\infty)\times[\rho,\infty)$, we use $K_R \le K$ to prove that

$$\begin{aligned}
T_2 &\le 12||\varphi||_\infty \int_{(\frac{R}{4},\infty)}\int_{[\rho,\infty)} K(\xi,z)\Phi_R(\mathrm{d}z)\Phi_R(\mathrm{d}\xi) \\
&\le c\left(R^{-\gamma-2\lambda}M^2_{\gamma+\lambda}(\Phi_R)+R^{\gamma+\lambda-1}\rho^{-\gamma-2\lambda}M_{\gamma+\lambda}(\Phi_R)\right),
\end{aligned}$$

which gives a small contribution due to the uniform estimates for Φ_R if we make R sufficiently large.

We now analyse (5.36). We consider a continuous function $g:\mathbb{R}_+\to[0,1]$ such that $g(x)=1$, when $x\le 1$, and $g(x)=0$, when $x\ge 2$. We define the function p_M as

$$p_M(x,y)=g\left(\frac{x}{M}\right)g\left(\frac{y}{M}\right), \tag{5.38}$$

where M is a positive constant. By the construction of p_M and given the fact that Φ_R is supported in the region $[\rho,\infty)$, for every $R>1$, we have that, given any function $\varphi\in C_c(\mathbb{R}_*)$,

$$\begin{aligned}
&\int_{[\rho,\infty)}\int_{[\rho,\infty)} K(\xi,z)p_M(\xi,z)\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)] \\
&\quad=\int_{[\rho,2M]}\int_{[\rho,2M]} K(\xi,z)p_M(\xi,z)\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)].
\end{aligned}$$

Therefore, since $\Phi_R\rightharpoonup\Phi$ in the weak-* topology, we can conclude that $\Phi_R\Phi_R\rightharpoonup\Phi\Phi$ in the weak-* topology as $R\to\infty$ if we work on the set $[\rho,2M]^2$ and we deduce that

$$\int_{[\rho,\infty)}\int_{[\rho,\infty)} K(\xi,z)p_M(\xi,z)\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)]$$

converges to

$$\int_{[\rho,\infty)}\int_{[\rho,\infty)} K(\xi,z)p_M(\xi,z)\Phi(\mathrm{d}\xi)\Phi(\mathrm{d}z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)]$$

as R tends to infinity.

To conclude that, for every test function $\varphi \in C_c(\mathbb{R}_*)$, we have that (5.36) converges to (5.37) as $R \to \infty$, we have to prove that the reminder terms, namely

$$\int_{[\rho,\infty)}\int_{[M,\infty)} K(\xi,z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)]\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z), \quad (5.39)$$

$$\int_{[M,\infty)}\int_{[\rho,\infty)} K(\xi,z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)]\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z), \quad (5.40)$$

$$\int_{[\rho,\infty)}\int_{[M,\infty)} K(\xi,z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)]\Phi(\mathrm{d}\xi)\Phi(\mathrm{d}z), \quad (5.41)$$

$$\int_{[M,\infty)}\int_{[\rho,\infty)} K(\xi,z)[\varphi(z+\xi)-\varphi(z)-\varphi(\xi)]\Phi(\mathrm{d}\xi)\Phi(\mathrm{d}z), \quad (5.42)$$

tend to zero as $M \to \infty$.

Let us look at the term (5.39). In this case we are in the region where $\{\xi \geq M\}$, hence $\xi^{\gamma+\lambda-1} \leq M^{\gamma+\lambda-1}$ as $\gamma+\lambda < 1$. Therefore, for every $\varphi \in \mathrm{C}_c(\mathbb{R}_{>0})$

$$\int_{[\rho,\infty)}\int_{[M,\infty)} \xi^{\gamma+\lambda}z^{-\lambda}\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)|\varphi(z+\xi)-\varphi(z)-\varphi(\xi)|$$
$$\lesssim M^{\gamma+\lambda-1}\rho^{-\gamma-2\lambda}M_{\gamma+\lambda}(\Phi_R),$$

where we remind that $M_{\gamma+\lambda}(\Phi_R)$ is bounded uniformly in $R > 1$ and that the mass of Φ_R is equal to one. Similarly, the fact that $\xi \geq M$ implies that $\xi^{-\gamma-2\lambda} \leq M^{-\gamma-2\lambda}$ since $\gamma + 2\lambda > 0$. We then obtain that

$$\int_{[\rho,\infty)}\int_{[M,\infty)} z^{\gamma+\lambda}\xi^{-\lambda}\Phi_R(\mathrm{d}\xi)\Phi_R(\mathrm{d}z)|\varphi(z+\xi)-\varphi(z)-\varphi(\xi)|$$
$$\leq CM^{-\gamma-2\lambda}M^2_{\gamma+\lambda}(\Phi_R).$$

From these two inequalities and the fact that $\gamma+\lambda < 1$ and $\gamma+2\lambda > 0$ we deduce that the term (5.39) tends to zero as $M \to \infty$. By a symmetric argument we prove that the term (5.40) tends to zero as $M \to \infty$. The fact that the two terms (5.41) and (5.42) tend to zero as $M \to \infty$ follows similarly by the fact that the $\gamma+\lambda$ moment of Φ is bounded.

The linear terms for the self-similar profiles of the truncated problem will converge as $R \to \infty$ to the desired terms thanks to (5.34).

To conclude, we need to prove the convergence of the remainder term. We thus have to prove that

$$\frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)}\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)$$

converges to

$$\frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x)}\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi(\mathrm{d}x)$$

as $R\to\infty$. We have that

$$\begin{aligned}
&\left|\frac{1}{M_{\gamma+\lambda}(\Phi_R)}\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) - \frac{1}{M_{\gamma+\lambda}(\Phi)}\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi(\mathrm{d}x)\right| \\
&\quad\le \left|\frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)}\right|\left|\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) - \int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi(\mathrm{d}x)\right| \\
&\qquad+\left|\frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)} - \frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x)}\right|\left|\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi(\mathrm{d}x)\right|.
\end{aligned}$$

We can analyse the two terms separately. For the first term, we know by (5.25) that the $M_{\gamma+\lambda}(\Phi_R)$ moment is uniformly bounded from below and thus

$$\begin{aligned}
&\left|\frac{1}{\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)}\right|\left|\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) - \int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi(\mathrm{d}x)\right| \\
&\le M_{2-\gamma-\lambda}(\Phi_R)\left|\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) - \int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi(\mathrm{d}x)\right| \\
&\le C_2\left|\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) - \int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi(\mathrm{d}x)\right|
\end{aligned}$$

and then we use that $\left|\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi_R(\mathrm{d}x) - \int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi(\mathrm{d}x)\right| \to 0$ by (5.34).

For the second term, we split the integral into a region with compact support and the remainder regions. More precisely,

$$\begin{aligned}
&\left|\frac{\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x) - \int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)}{\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x)}\right|\left|\int_{[\rho,\infty)} \partial_x\varphi(x)x^{\gamma+\lambda}\Phi(\mathrm{d}x)\right| \\
&\le C\left|\int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x) - \int_{[\rho,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)\right| M_{2-\gamma-\lambda}(\Phi_R)M_{2-\gamma-\lambda}(\Phi)M_{\gamma+\lambda}(\Phi) \\
&\lesssim C\left|\int_{[\rho,\infty)} g\left(\frac{x}{M}\right)\left[x^{\gamma+\lambda}\Phi(\mathrm{d}x) - x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)\right]\right| + 2C\left|\int_{[M,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x)\right| \\
&\quad + 2C\left|\int_{[M,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)\right|.
\end{aligned}$$

We have that $\left|\int_{[\rho,\infty)} g(\frac{x}{M})[x^{\gamma+\lambda}\Phi(\mathrm{d}x) - x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)]\right| \to 0$ as $R \to \infty$ by (5.34). For the regions close to infinity, we use

$$\left|\int_{[M,\infty)} x^{\gamma+\lambda}\Phi_R(\mathrm{d}x)\right| \leq M^{\gamma+\lambda-1},$$

as the total mass is uniformly bounded. Since $\gamma+\lambda<1$, the remaining term goes to zero as $M\to\infty$.

As a last step we prove that, if $\Phi((0,\delta))=0$ for some $\delta>0$, then $\Phi((0,\rho(M_{\gamma+\lambda})))=0$ for $\rho(M_{\gamma+\lambda})$ given by (3.4). Since the proof is standard we only sketch it here. Consider $x\in[\frac{\delta}{2},\frac{1}{2}\rho(M_{\gamma+\lambda}))$, then, since Φ satisfies (3.1), we have

$$\begin{aligned} c\Phi(x) &\leq \left(\frac{x^{\gamma+\lambda}}{M_{\gamma+\lambda}} - \frac{2}{1-\gamma}x\right)\Phi(x) \\ &\leq \frac{|1+\gamma|}{1-\gamma}\int_\delta^x \Phi(\eta)\mathrm{d}\eta + \int_\delta^x\int_\delta^{y-\delta} K(y-\eta,\eta)\Phi(y-\eta)\Phi(\eta)\mathrm{d}\eta\mathrm{d}y \\ &\leq \tilde{c}\int_\delta^x \Phi(\eta)\mathrm{d}\eta, \end{aligned}$$

for some positive constants $\tilde{c}, c>0$. Using Grönwall's lemma, this implies that $\Phi(x) \leq e^{\frac{\tilde{c}}{c}x}\Phi(\frac{\delta}{2}) = 0$. Hence $\Phi([\delta,\frac{1}{2}\rho(M_{\gamma+\lambda}))) = 0$. Iterating this argument, we deduce that $\Phi((0,\rho(M_{\gamma+\lambda})))=0$. The argument can be made rigorous by working with the weak formulation of Eq. (3.1). □

5.4. Properties of the constructed self-similar profile

In this section we aim at proving Theorem 3.3. In particular we prove that the self-similar profile, whose existence has been proven in Theorem 5.1, satisfies the properties stated in Theorem 3.3. In other words we have to prove the following theorem.

Theorem 5.13. *Assume K to be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (1.5), (1.6)*, with γ,λ such that* (1.7) *holds and such that*

$$-1<\gamma, \quad \gamma+2\lambda\geq 1.$$

Let Φ be the self-similar profile constructed in Theorem 5.1*. Then*

$$\int_{\mathbb{R}_*} e^{Lx}\Phi(\mathrm{d}x) < \infty$$

for some $L>0$ and Φ is absolutely continuous with respect to the Lebesgue measure. Then $\Phi(\mathrm{d}x)=\phi(x)\mathrm{d}x$ and ϕ is such that

$$\limsup_{x\to\infty} e^{\overline{L}x}\phi(x) < \infty$$

for a constant $\overline{L}>0$.

To prove this theorem we start by proving that every self-similar profile Φ in the sense of Definition 3.1 and such that $\Phi((0,\delta)) = 0$, for a positive δ, is absolutely continuous with respect to the Lebesgue measure and satisfies some moment bounds.

Proposition 5.14. (Regularity) *Assume K to be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (1.5), (1.6)*, with γ, λ such that* (1.7) *holds and such that $\gamma + 2\lambda \geq 1$. Let $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ be a self-similar profile as in Definition 3.1. Assume additionally that $\Phi((0,\delta)) = 0$ for $\delta > 0$. Then Φ is absolutely continuous with respect to the Lebesgue measure. Its density $\phi \in L^1([\delta,\infty))$ satisfies the following equation*

$$\begin{aligned}\int_0^z x\phi(x)dx - \frac{2}{1-\gamma}z^2\phi(z) + \frac{1}{M_{\gamma+\lambda}(\phi)}z^{\gamma+\lambda+1}\phi(z)\\ - \frac{1}{M_{\gamma+\lambda}(\phi)}\int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x\\ = -J_\phi(z), \quad a.e.\ z > \delta\end{aligned} \tag{5.43}$$

where

$$J_\phi(z) := \int_0^z \int_{z-x}^\infty xK(x,y)\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x. \tag{5.44}$$

Proof. We just sketch the proof as similar arguments have been repeatedly used in the analysis of coagulation equations, see for instance the proof of Lemma 4.9 in [9]. An analogous proof will be presented in the proof of Proposition 6.2. Let Φ be a solution of (3.1), then

$$\partial_\xi\left[\left(\frac{\xi^{\gamma+\lambda}}{M_{\gamma+\lambda}} - \frac{2}{1-\gamma}\xi\right)\Phi(\xi)\right] = \frac{1+\gamma}{1-\gamma}\Phi(\xi) + \mathbb{K}[\Phi](\xi).$$

Using the fact that

$$\mathbb{K}[\Phi](\xi) \leq \int_\delta^{\xi-\delta} K(\xi-\eta,\eta)\Phi(\eta)\Phi(\xi-\eta)\mathrm{d}\eta, \quad \xi > \delta$$

and that $\int_\delta^{\xi-\delta} K(\xi-\eta,\eta)\Phi(\eta)\Phi(\xi-\eta)\mathrm{d}\eta$ is a bounded measure in every compact set, we deduce that the measure

$$\left(\frac{\xi^{\gamma+\lambda}}{M_{\gamma+\lambda}} - \frac{2}{1-\gamma}\xi\right)\Phi(\xi)$$

is absolutely continuous with respect to the Lebesgue measure on the set (δ,∞) if $\delta > \rho(M_{\gamma+\lambda})$, while it is absolutely continuous with respect to the Lebesgue measure in the set $(\delta, \rho(M_{\gamma+\lambda})) \cup (\rho(M_{\gamma+\lambda}),\infty)$ if $\delta \leq \rho(M_{\gamma+\lambda})$. In the latter case, we therefore have that $\Phi(\mathrm{d}x) = \phi(x)\mathrm{d}x + c_0\delta_{\rho(M_{\gamma+\lambda})}(x)$ with density $\phi \in$

$L^1(\mathbb{R}_*)$ and with $c_0 \in \mathbb{R}$. To exclude that the self-similar solution Φ has a Dirac in $\rho(M_{\gamma+\lambda})$, we notice that

$$\int_{\delta}^{\xi-\delta} K(\xi-\eta,\eta)c_0\delta_{\rho(M_{\gamma+\lambda})}(\eta)c_0\delta_{\rho(M_{\gamma+\lambda})}(\xi-\eta)\mathrm{d}\eta$$
$$= c_0^2\int_{\delta}^{\xi-\delta} K(\xi-\eta,\eta)\delta_{\rho(M_{\gamma+\lambda})}(\eta)\delta_{\rho(M_{\gamma+\lambda})}(\xi-\eta)\mathrm{d}\eta$$

is non-zero only if $\xi = 2\rho(M_{\gamma+\lambda})$. Hence the coagulation operator applied to measure $c_0\delta_{\rho(M_{\gamma+\lambda})}$ produces a Dirac in $2\rho(M_{\gamma+\lambda})$, contradicting the fact that $\Phi_{|(\rho(M_{\gamma+\lambda}),\infty)}$ is absolutely continuous with respect to the Lebesgue measure. As a consequence, we deduce that $c_0 = 0$. □

Proposition 5.15. (Moment bounds) *Assume K to be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (1.5), (1.6)*, with γ,λ such that* (1.7) *holds and such that $\gamma+2\lambda \geq 1$. Let $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ be a self-similar profile as in Definition 3.1 and assume additionally that $\Phi((0,\delta)) = 0$, where $\delta > 0$. Then*

$$\int_{\mathbb{R}_*} x^{\mu}\Phi(\mathrm{d}x) = \int_{[\delta,\infty)} x^{\mu}\Phi(\mathrm{d}x) < \infty, \quad \forall \mu \in \mathbb{R}.$$

Proof. For $\mu \leq \gamma+\lambda$ the statement follows by the fact that $\Phi((0,\delta)) = 0$ and that by definition of self-similar profile we know that

$$\int_{[\delta,\infty)} x^{\gamma+\lambda}\Phi(\mathrm{d}x) < \infty.$$

We want to prove the statement for $\mu > \gamma+\lambda$. As a first step, we prove that there exists a $\overline{\delta} > 0$ such that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda+\overline{\delta}}\phi(x)\mathrm{d}x < \infty, \tag{5.45}$$

where ϕ is the density of Φ. As a second step we will prove that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda+n\overline{\delta}}\phi(x)\mathrm{d}x < \infty \tag{5.46}$$

for $n \geq 1$ implies that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda+(n+1)\overline{\delta}}\phi(x)\mathrm{d}x < \infty. \tag{5.47}$$

The desired conclusion then follows by induction.

Step 1

Consider an integrable function φ and integrate both sides of Eq. (5.43) against the function $\psi(x) = \int_x^\infty \varphi(y)\mathrm{d}y$ to deduce that

$$\int_0^\infty \psi(z) z\phi(z)\mathrm{d}z - \int_0^\infty \phi(z)\psi'(z)\left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2}{1-\gamma}z^2\right)\mathrm{d}z - \frac{1}{M_{\gamma+\lambda}}\int_0^\infty \psi(z) z^{\gamma+\lambda}\phi(z)\mathrm{d}z = \int_0^\infty\int_0^\infty xK(x,y)\left(\psi(x+y)-\psi(x)\right)\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x. \tag{5.48}$$

We remark that the above formulation requires that ψ has to decay fast enough so that all the integrals in (5.48) are finite.

Select $\overline{\delta} > 0$ such that $\max\{0, 1-2(\gamma+\lambda)\} < \overline{\delta} < 1-\gamma-\lambda$ and consider $\psi_R(x) = x^{\gamma+\lambda+\overline{\delta}-1}$ if $x \leq R$ while $\psi_R(x) = R^{\overline{\delta}} x^{\gamma+\lambda-1}$ if $x > R$. Using (5.48) and the fact that $\gamma + 2\lambda \geq 1$ and that $\psi_R(x+y) - \psi_R(x) \leq 0$ we deduce that

$$0 < \frac{\gamma+2\lambda-1+2\overline{\delta}}{1-\gamma}\int_0^R z^{\gamma+\lambda+\overline{\delta}}\phi(z)\mathrm{d}z \leq \frac{1}{M_{\gamma+\lambda}}\int_0^\infty z^{2(\gamma+\lambda)+\overline{\delta}-1}\phi(z)\mathrm{d}z < \infty,$$

where the moment $M_{2(\gamma+\lambda)+\overline{\delta}-1}$ is bounded because, due to the choice of $\overline{\delta}$, we have that $2(\gamma+\lambda)+\overline{\delta}-1 < \gamma+\lambda$. Passing to the limit as R tends to infinity we deduce that (5.45) holds.

Step 2

We assume that inequality (5.46) holds and we want to prove (5.47). Taking $\psi_R(x) = x^{\gamma+\lambda+(n+1)\overline{\delta}-1}$ when $x \leq R$ and $\psi_R(x) = R^{\overline{\delta}} x^{\gamma+\lambda+n\overline{\delta}-1}$ when $x > R$ in (5.48) we deduce that

$$\begin{aligned} 0 &< \frac{\gamma+2\lambda-1+2\overline{\delta}(n+1)}{1-\gamma}\int_0^R z^{\gamma+\lambda+(n+1)\overline{\delta}}\phi(z)\mathrm{d}z \\ &\leq \frac{1}{M_{\gamma+\lambda}}\int_0^\infty z^{2(\gamma+\lambda)+(n+1)\overline{\delta}-1}\phi(z)\mathrm{d}z \\ &\quad + \int_\delta^\infty\int_\delta^\infty xK(x,y)\left[\psi_R(x+y)-\psi_R(x)\right]\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \\ &\quad + \int_0^\infty \phi(z)\psi_R'(z) z^{\gamma+\lambda+1}\phi(z)\mathrm{d}z. \end{aligned}$$

Thanks to the choice of $\overline{\delta}$, we have that $2(\gamma+\lambda)+(n+1)\overline{\delta}-1 \leq \gamma+\lambda+n\overline{\delta}$ and the desired conclusion follows for n such that $\gamma+\lambda+(n+1)\overline{\delta}-1 < 1$ by the fact that in that case ψ_R is decreasing and hence the coagulation term is negative as well as the term $\int_0^\infty \phi(z)\psi_R'(z)z^{\gamma+\lambda+1}\phi(z)$. We examine now the case in which

$\gamma + \lambda + (n+1)\overline{\delta} - 1 \geq 1$. In this case we have that

$$
\begin{aligned}
&\int_\delta^\infty \int_\delta^\infty xK(x,y)\left[\psi_R(x+y) - \psi_R(x)\right]\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \\
&\leq c \int_\delta^\infty \int_\delta^\infty xK(x,y)\Big[(x+y)^{\gamma+\lambda+(n+1)\overline{\delta}-1} \\
&\quad -x^{\gamma+\lambda+(n+1)\overline{\delta}-1}\Big]\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \\
&\leq c \iint_{\{(x,y)\in[\delta,\infty)^2 : x\leq y\}} x^{1-\lambda}y^{\gamma+\lambda}\Big[(x+y)^{\gamma+\lambda+(n+1)\overline{\delta}-1} \\
&\quad -x^{\gamma+\lambda+(n+1)\overline{\delta}-1}\Big]\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \\
&\quad + c \iint_{\{(x,y)\in[\delta,\infty)^2 : x\geq y\}} x^{1+\gamma+\lambda}y^{-\lambda}\Big[(x+y)^{\gamma+\lambda+(n+1)\overline{\delta}-1} \\
&\quad -x^{\gamma+\lambda+(n+1)\overline{\delta}-1}\Big]\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \\
&\leq c \int_\delta^\infty x^{1-\lambda}\phi(x)\mathrm{d}x \int_\delta^\infty y^{2(\gamma+\lambda)+(n+1)\overline{\delta}-1}\phi(y)\mathrm{d}y < \infty.
\end{aligned}
$$

Since we also have that

$$
\int_0^\infty \phi(z)\psi_R'(z)z^{\gamma+\lambda+1}\phi(z)\mathrm{d}z \leq c(n)\int_0^\infty \phi(z)z^{2(\gamma+\lambda)+(n+1)\delta-1}\phi(z)\mathrm{d}z < \infty
$$

the desired conclusion follows. □

Lemma 5.16. (Exponential bound) *Assume K to be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying (1.5), (1.6), with γ, λ such that (1.7) holds and such that $\gamma + 2\lambda \geq 1$. Let $\Phi \in \mathcal{M}_+(\mathbb{R})$ be as in Proposition 5.14 and let ϕ be its density. Then there exist two positive constants L and M such that*

$$
\int_M^\infty \phi(z)e^{Lz}\mathrm{d}z < \infty. \tag{5.49}
$$

Proof. Adapting the approach used in [11] we define the function Ψ_a as

$$
\Psi_a(L) := \int_M^\infty \frac{e^{L\min\{x,a\}}}{\min\{x,a\}}x^2\phi(x)\mathrm{d}x
$$

where $M > \delta$. Hence, by its definition

$$
\Psi_a'(L) = \int_M^\infty e^{L\min\{x,a\}}x^2\phi(x)\mathrm{d}x.
$$

We consider a function ψ in (5.48) with $\psi'(x) := e^{L\min\{x,a\}}$ to deduce that

$$\begin{aligned}\frac{2}{1-\gamma}\Psi_a'(L) \leq & \frac{M^{\gamma+\lambda-1}}{M_{\gamma+\lambda}}\int_M^\infty e^{L\min\{x,a\}}x^2\phi(x)\mathrm{d}x \\ &+ \frac{1}{M_{\gamma+\lambda}}\int_\delta^M e^{L\min\{x,a\}}x^{\gamma+\lambda+1}\phi(x)\mathrm{d}x \\ &+ \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} xK(x,y)\int_y^{x+y} e^{L\min\{w,a\}}\mathrm{d}w\phi(x)\phi(y)\mathrm{d}x\mathrm{d}y \\ &+ \int_\delta^{\overline{z}}\left(\frac{1}{M_{\gamma+\lambda}}x^{\gamma+\lambda}-x\right)\psi(x)\phi(x)\mathrm{d}x,\end{aligned}$$

for $\overline{z} := \left(\frac{1}{M_{\gamma+\lambda}}\right)^{\frac{1}{1-\gamma-\lambda}}$, where we are assuming without loss of generality that $\overline{z} > \delta$ and since for every $z \geq \overline{z}$ we have

$$z - \frac{z^{\gamma+\lambda}}{M_{\gamma+\lambda}} > 0. \tag{5.50}$$

Thus, there exists a $c \geq 0$ such that

$$\begin{aligned}\frac{2}{1-\gamma}\Psi_a'(L) \leq & \Psi_a'(L)\frac{M^{\gamma+\lambda-1}}{M_{\gamma+\lambda}} + c \\ &+ \int_{\mathbb{R}_*}\int_{\mathbb{R}_*} xK(x,y)\int_y^{x+y} e^{L\min\{w,a\}}dw\phi(x)\phi(y)dxdy.\end{aligned}$$

As in [9,11] we can deduce, using Jensen's inequality together with the fact that $\gamma+\lambda<1$ and that $-\lambda<1$, that

$$\int_{\mathbb{R}_*}\int_{\mathbb{R}_*} xK(x,y)\int_y^{x+y} e^{L\min\{w,a\}}\mathrm{d}w\phi(x)\phi(y)\mathrm{d}x\mathrm{d}y \leq \Psi_a(L)^{1-\gamma-\lambda}\Psi_a'(L)^{\gamma+\lambda}.$$

This, together with the fact that we can take M arbitrary large, implies that

$$c(\gamma,\lambda)\Psi_a'(L) \leq \Psi_a(L)^{1-\gamma-\lambda}\Psi_a'(L)^{\gamma+\lambda} + c, \tag{5.51}$$

for a positive constant c and for $c(\gamma,\lambda) = \frac{2}{1-\gamma} - \frac{M^{\gamma+\lambda-1}}{M_{\gamma+\lambda}} > 0$.

By the definition of Ψ_a we have that

$$\Psi_a(0) \leq \frac{2}{1-\gamma}\left(1+\frac{c_1}{a}\right) \to \frac{2}{1-\gamma} \text{ as } a\to\infty.$$

If we prove that $\limsup_{a\to\infty}\Psi_a(L) < \infty$ for some L, then we can conclude. Indeed we would have

$$M\int_M^\infty e^{Lx}\phi(x)\mathrm{d}x \leq \int_M^\infty e^{Lx}x\phi(x)\mathrm{d}x < \infty.$$

Let us prove that $\limsup_{a\to\infty} \Psi_a(L) < \infty$ for some L. First of all notice that this is true if $\Psi'_a(L) \leq 1$ as in this case $\Psi_a(L) \leq L + \lim_{a\to\infty} \Psi_a(0)$ and the desired conclusion follows. If instead $\Psi'_a > 1$, then

$$c(\gamma,\lambda)\Psi'_a(L) \leq \Psi_a(L)^{1-\gamma-\lambda}\Psi'_a(L)^{\gamma+\lambda} + c \leq \alpha\Psi_a(L)^{1-\gamma-\lambda}\Psi'_a(L) + c.$$

Without loss of generality we can assume $\alpha > 0$ such that $\lim_{a\to\infty} \Psi_a(0) \neq \alpha$. This implies that a solution of the ODE

$$\Psi'_a\left(1 - \alpha\Psi_a^{1-\gamma-\lambda}\right) = c$$

exists for small intervals around 0 and is such that $\lim_{a\to\infty} \Psi_a(L) < \infty$ for L in that interval. □

Proposition 5.17. (Exponential decay) *Assume K to be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (1.5), (1.6)*, with γ, λ such that* (1.7) *holds and such that $\gamma + 2\lambda \geq 1$. Let $\Phi \in \mathcal{M}_+(\mathbb{R})$ be as in Proposition 5.14 and let ϕ be its density. Then there exists a positive constant $\tilde{M}$ such that*

$$\limsup_{z\to\infty} \phi(z)e^{\tilde{M}z} < \infty. \tag{5.52}$$

Proof. From Eq. (5.43) we deduce that

$$\begin{aligned}&\left(\frac{2}{1-\gamma}z^2 - \frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}(\phi)}\right)\phi(z)\\&\quad\leq \int_0^z x\phi(x)\mathrm{d}x - \frac{1}{M_{\gamma+\lambda}(\phi)}\int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x + J_\phi(z).\end{aligned}$$

We now show that J_ϕ decays exponentially and that the term

$$\int_0^z x\phi(x)\mathrm{d}x - \frac{1}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x$$

decays exponentially too.

First of all, we prove that there exists a constant $M_1 > 0$ such that $J_\phi(z) \leq e^{-M_1 z}$ for large z. To this end we notice that the bound (5.49) implies that

$$\begin{aligned}J_\phi(z) &= \int_0^z\int_{z-x}^\infty e^{-L(x+y)}e^{L(x+y)}xK(x,y)\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x\\&\leq e^{-Lz}\int_0^z\int_{z-x}^\infty e^{L(x+y)}xK(x,y)\phi(x)\phi(y)\mathrm{d}y\mathrm{d}x \leq e^{-M_1 z}.\end{aligned}$$

On the other side, for large values of z we have that

$$
\begin{aligned}
&\int_0^z x\phi(x)\mathrm{d}x - \frac{1}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x \\
&\quad = \frac{1}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\left(\int_0^z x\phi(x)\mathrm{d}x\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y - \int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x\right) \\
&\quad \leq \frac{1}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\left(\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y - \int_0^z x^{\gamma+\lambda}\phi(x)\mathrm{d}x\right) \\
&\quad \leq \frac{1}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\int_z^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y \leq C(\rho(M_{\gamma+\lambda}))e^{-M_2 z},
\end{aligned}
$$

where $M_2 > 0$.

We deduce that for large values of z

$$
\left(\frac{2z^2}{1-\gamma} - \frac{z^{\gamma+\lambda+1}}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y}\right)\phi(z) \leq c\max\{e^{-zM_1}, e^{-zM_2}\}.
$$

The conclusion follows since, for every $z > \left(\frac{1-\gamma}{2\int_0^\infty x^{\gamma+\lambda}\Phi(\mathrm{d}x)}\right)^{\frac{1}{1-\gamma-\lambda}}$, we have

$$
\frac{2z^2}{1-\gamma} - \frac{z^{\gamma+\lambda+1}}{\int_0^\infty y^{\gamma+\lambda}\phi(y)\mathrm{d}y} > 0.
$$

□

6. Non-existence results and properties of the self-similar profiles

To study the non-existence of the self-similar solutions we proceed by contradiction and start by assuming that a self-similar solution exists. To find a contradiction we need to analyse the properties of each self-similar profile. In the case $\gamma + 2\lambda > 1$ the fundamental properties that we prove to be true are that $\Phi((0, \delta)) = 0$ for some positive $\delta > 0$ and that Φ decays sufficiently fast for large sizes. When $\gamma + 2\lambda = 1$, it is possible to prove that Φ decays fast for large sizes, but we have not been able to prove that $\Phi((0, \delta)) = 0$ for some $\delta > 0$. This is the reason why we require the additional condition (3.5) in Theorem 3.5.

6.1. Properties of the self-similar profiles

In this section we study the properties of each self-similar solution that do not rely on the existence of the self-similar solution and hence do not rely on the assumption $\gamma > -1$.

Theorem 6.1. *(Properties of the self-similar profiles) Let K be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying* (1.5), (1.6)*, with γ, λ such that* (1.7) *holds.*

1. *If $\gamma + 2\lambda > 1$, then every self-similar profile Φ as in Definition 3.1 is such that $\Phi((0, \rho(M_{\gamma+\lambda}))) = 0$ where $\rho(M_{\gamma+\lambda})$ is given by (3.4). Additionally, Φ is such that*

$$\int_{\mathbb{R}_*} e^{Lx} \Phi(\mathrm{d}x) < \infty \tag{6.1}$$

for $L > 0$ and it is absolutely continuous with respect to the Lebesgue measure. Its density ϕ is such that

$$\limsup_{x \in \mathbb{R}_*} \phi(x) e^{\overline{L}x} \leq c \tag{6.2}$$

for constants $\overline{L}, c > 0$.

2. *If $\gamma + 2\lambda = 1$ and if, in addition, $\Phi((0, \delta)) = 0$ for some $\delta > 0$, then $\Phi([\delta, \rho(M_{\gamma+\lambda}))) = 0$ where $\rho(M_{\gamma+\lambda})$ is given by (3.4). In addition, (6.1) and (6.2) hold.*

First of all we prove, see Proposition 6.2, that when $\gamma + 2\lambda > 1$, each solution of Eq. (1.19) in the sense of Definition 3.1 is zero near the origin. The statement of Theorem 6.1 then follows by Proposition 5.14 and Proposition 5.17 and by the fact that when $\gamma + \lambda = 1$ we are assuming that there exists a $\delta > 0$ such that $\Phi((0, \delta)) = 0$.

Proposition 6.2. (Support of the self-similar solution) *Let K be a homogeneous symmetric coagulation kernel, of homogeneity γ, satisfying (1.5), (1.6), with γ, λ such that (1.7) holds and such that $\gamma + 2\lambda > 1$. Let $\Phi \in \mathcal{M}_+(\mathbb{R}_*)$ be a self-similar profile as in Definition 3.1. Then $\Phi((0, \overline{\xi})) = 0$ for*

$$\overline{\xi} := \min \left\{ \left(\frac{1}{M_{\gamma+\lambda}(\Phi)} \right)^{\frac{1}{1-\gamma-\lambda}}, \left(\frac{1-\gamma}{2M_{\gamma+\lambda}(\Phi)} \right)^{\frac{1}{1-\gamma-\lambda}} \right\}.$$

Proof. We start explaining the proof in a heuristic manner without entering in the technical details that will be explained later. Equation (1.19) can be rewritten in the following flux form

$$\partial_x \left(J_\Phi(x) + \left(\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2x^2}{1-\gamma} \right) \Phi(x) \right) = \left(\frac{x^{\gamma+\lambda}}{M_{\gamma+\lambda}} - x \right) \Phi(x), \tag{6.3}$$

where J_ϕ is given by (5.44). For x smaller than $\overline{\xi}$ we have that $\frac{x^{\gamma+\lambda}}{M_{\gamma+\lambda}} - x \geq 0$ and hence the function

$$x \mapsto J_\Phi(x) + \left(\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2x^2}{1-\gamma} \right) \Phi(x)$$

is increasing and right-continuous at $x = 0$. This implies that

$$\lim_{x \to 0^+} \left(J_\Phi(x) + \left(\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2x^2}{1-\gamma} \right) \Phi(x) \right) = L.$$

Since for every $x \leq \overline{\xi}$ we also have that $\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2x^2}{1-\gamma} \geq 0$ we deduce that $L \geq 0$.

We make a scaling argument to identify the value of L. To this end we notice that the units of the flux are given by $[J_\Phi(x)] = [\Phi]^2[x]^{3+\gamma}$, where from now on we indicate with $[\cdot]$ the dimensional properties of a quantity, hence

$$[\Phi] = [x]^{-\frac{(3+\gamma)}{2}}.$$

We deduce that

$$[x^{\gamma+\lambda+1}\Phi] = [x^{\gamma+\lambda+1}][\Phi] = [x^{\frac{\gamma+2\lambda-1}{2}}].$$

Since $\gamma + 2\lambda > 1$ this implies that as $x \to 0$ the dominant term in Eq. (6.3) is J_Φ, hence $J_\Phi(x) \sim L$ as $x \to 0$. Finally, we prove that $L = 0$, in agreement with the statement proven in [10] that, when $\gamma + 2\lambda \geq 1$, there are no solutions to the constant flux equation. Integrating (6.3), we deduce that

$$J_\Phi(x) + \left(\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2x^2}{1-\gamma}\right)\Phi(x) = \int_0^x \left(\frac{z^{\gamma+\lambda}}{M_{\gamma+\lambda}} - z\right)\Phi(z)\mathrm{d}z.$$

A detailed analysis of this ODE for Φ implies that $\Phi = 0$ on the interval $\left(0, \left(\frac{1-\gamma}{2M_{\gamma+\lambda}}\right)^{\frac{1}{1-\gamma-\lambda}}\right)$, see Step 4.

We now explain the proof in detail. Testing (3.3) with a function of the form $\psi(\xi) = \xi\varphi(\xi)$, where $\varphi \in \mathrm{C}^1_c(\mathbb{R}_*)$, we get to the following equation:

$$\int_{(0,\infty)} \left(\xi - \frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}\right)\varphi(\xi)\Phi(\mathrm{d}\xi) = \int_{(0,\infty)} \varphi'(\xi)W[\Phi](\mathrm{d}\xi), \tag{6.4}$$

where

$$W[\Phi](\mathrm{d}x) = J_\Phi(x)\mathrm{d}x + \left(\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2x^2}{1-\gamma}\right)\Phi(\mathrm{d}x). \tag{6.5}$$

Step 1: Regularity of $W[\Phi]$ and an integral representation formula
First of all we prove that, for every $\overline{\delta} > 0$, the restriction of the measure $W[\Phi]$ on the interval $[\overline{\delta}, 1]$ is absolutely continuous with respect to the Lebesgue measure. This follows from the fact that for every function $\varphi \in C^1_c([\overline{\delta}, \overline{\xi}])$ with $0 < \overline{\delta} < 1$ we have that

$$\int_{\mathbb{R}_*} \varphi'(\xi)W[\Phi](\mathrm{d}\xi) \leq c(\Phi, \delta, \gamma, \lambda)\|\varphi\|_\infty;$$

hence $W[\Phi]$, on the interval $[\overline{\delta}, \overline{\xi}]$, has a density $W_\phi \in BV([\overline{\delta}, \overline{\xi}])$, see [1]. Since $\overline{\delta}$ is arbitrary we deduce that $W[\Phi]$ has a density $W_\phi \in L^1_{loc}((0, \overline{\xi}])$. This implies that the measure $\Phi_{|(0,\overline{\xi}]}$ is absolutely continuous with respect to the Lebesgue measure, its density is ϕ.

We now prove that W_ϕ is increasing on $(0, \overline{\xi}]$. To this end we consider a sequence of functions $\{\varphi_n\}_n \subset C^1_c(\mathbb{R}_*)$ such that $\varphi_n \to \chi_{[\xi_1,\xi_2]}$ pointwise, with $0 < \xi_1, \xi_2 <$

$\overline{\xi}$, $\varphi_n'(\xi) \geq 0$ for $\xi \in \left(0, \frac{\xi_1+\xi_2}{2}\right)$ and such that $\varphi_n'(\xi) \leq 0$ for $\xi \in \left(\frac{\xi_1+\xi_2}{2}, \infty\right)$. Substituting φ_n in (6.4) we deduce that for every $n \geq 1$

$$\int_0^\infty \xi\varphi_n(\xi)\phi(\xi)\mathrm{d}\xi - \frac{1}{M_{\gamma+\lambda}}\int_0^\infty \varphi_n(\xi)\xi^{\gamma+\lambda}\phi(\xi)\mathrm{d}\xi = \int_0^\infty \varphi_n'(\xi)W_\phi(\xi)\mathrm{d}\xi.$$

Notice that

$$\int_0^\infty \xi\varphi_n(\xi)\phi(\xi)\mathrm{d}\xi \to \int_{[\xi_1,\xi_2]} \xi\phi(\xi)\mathrm{d}\xi \quad \text{as} \quad n \to \infty$$

and that

$$\int_0^\infty \xi^{\gamma+\lambda}\varphi_n(\xi)\phi(\xi)\mathrm{d}\xi \to \int_{[\xi_1,\xi_2]} \xi^{\gamma+\lambda}\phi(\xi)\mathrm{d}\xi \quad \text{as} \quad n \to \infty.$$

Additionally, since W_ϕ is in $L^1_{loc}((0,\overline{\xi}))$ and satisfies (6.4) we deduce that $W_\phi \in W^1_{loc}(0,\overline{\xi})$ and hence is continuous in $(0,\overline{\xi})$. Hence we have that

$$\int_{(0,\infty)} \varphi_n'(\xi)W_\phi(\xi)\mathrm{d}\xi \to W_\phi(\xi_1) - W_\phi(\xi_2) \quad \text{as} \quad n \to \infty.$$

As a consequence, we deduce that

$$W_\phi(\xi_2) - W_\phi(\xi_1) = \int_{[\xi_1,\xi_2]} \left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \xi\right)\phi(\xi)\mathrm{d}\xi. \tag{6.6}$$

Since if $\xi \leq \overline{\xi}$, then $\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \xi \geq 0$, we deduce that there exists a $\overline{\xi}$ such that W_ϕ is increasing in $(0,\overline{\xi}]$. Hence, since W_ϕ is bounded, we have that the following limit exists

$$0 \leq L := \lim_{\xi\to 0^+} W_\phi(\xi) < \infty.$$

Due to (3.2) we can take the limit as $\xi_1 \to 0$ in (6.6) and deduce that

$$W_\phi(\xi_2) - L = \int_{(0,\xi_2]} \left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \xi\right)\phi(\xi)\mathrm{d}\xi. \tag{6.7}$$

Step 2: we prove that $\int_{\mathbb{R}_*} J_\phi(\varepsilon x)\varphi(x)\mathrm{d}x \to L\int_{\mathbb{R}_*}\varphi(x)\mathrm{d}x$
Since W_ϕ is bounded in $(0,\overline{\xi}]$ and $\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \frac{2}{1-\gamma}\xi \geq 0$ for every $\xi \leq \overline{\xi}$ we have that J_ϕ is also bounded in $(0,\overline{\xi}]$. Since

$$[2/3z, z] \times [2/3z, z] \subset \left\{x \in \mathbb{R}^2_* : 0 < x \leq z, z - x \leq y < \infty\right\} =: \Omega_z \tag{6.8}$$

we deduce that for every $z \in (0,\overline{\xi}]$ we have that

$$z^{\gamma+1}\left(\int_{2z/3}^z \phi(x)\mathrm{d}x\right)^2 \leq J_\phi(z) \leq C,$$

hence

$$\frac{1}{z}\int_{2z/3}^{z}\phi(x)\mathrm{d}x \leq \frac{C}{z^{\frac{\gamma+3}{2}}} \quad z\in(0,\overline{\xi}]. \tag{6.9}$$

Equation (6.7) implies that

$$\begin{aligned}\int_{\mathbb{R}_*}\varphi(x)\,J_\phi(\varepsilon x)\mathrm{d}x &= \frac{1}{\varepsilon}\int_{\mathbb{R}_*}\varphi\left(\frac{x}{\varepsilon}\right)J_\phi(x)\mathrm{d}x\\ &= -\frac{1}{\varepsilon}\int_{\mathbb{R}_*}\varphi\left(\frac{x}{\varepsilon}\right)\left[\frac{x^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2}{1-\gamma}x^2\right]\phi(x)\mathrm{d}x \\ &+\frac{1}{\varepsilon}\int_{\mathbb{R}_*}\varphi\left(\frac{x}{\varepsilon}\right)\int_0^x\left[\frac{z^{\gamma+\lambda}}{M_{\gamma+\lambda}}-z\right]\phi(z)\mathrm{d}z\mathrm{d}x+L\int_{\mathbb{R}_*}\varphi(x)\mathrm{d}x\end{aligned} \tag{6.10}$$

where $\varphi\in C_c((0,\overline{\xi}))$ and where $\varepsilon>0$.

Thanks to the bounds (3.2) for ϕ and to the fact that φ is compactly supported we can pass to the limit as $\varepsilon\to 0$ in Eq. (6.10) and deduce that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_*}J_\phi(\varepsilon\xi)\varphi(\xi)\mathrm{d}\xi = L\int_{\mathbb{R}_*}\varphi(x)\mathrm{d}x$$

for every $\varphi\in C_c^1((0,\overline{\xi}))$.

Step 3: we prove that $L=0$
We want to prove that for every $\varphi\in C_c^1((0,\overline{\xi}))$

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_*}J_\phi(\varepsilon\xi)\varphi(\xi)\mathrm{d}\xi=0.$$

Using the formula for the fluxes we obtain that

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\mathbb{R}_*}J_\phi(\xi)\varphi\left(\frac{\xi}{\varepsilon}\right)\mathrm{d}\xi &= \int_{\mathbb{R}_*}\varphi(\xi)J_\phi(\varepsilon\xi)\mathrm{d}\xi\\ &=\int_{\mathbb{R}_*}\int_{\mathbb{R}_*}xK(x,y)\left(\psi\left(\frac{x+y}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}\right)\right)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\ &= I_1+I_2\end{aligned}$$

where $\psi(x)=\int_0^x\varphi(y)\mathrm{d}y$ and where

$$I_1=\int_{\mathbb{R}_*}\int_{(y,\infty)}xK(x,y)\left(\psi\left(\frac{x+y}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}\right)\right)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)$$

and

$$I_2=\int_{\mathbb{R}_*}\int_{(0,y)}xK(x,y)\left(\psi\left(\frac{x+y}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}\right)\right)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y).$$

Since φ is compactly supported we have, by definition, that $\operatorname{supp}[\psi(x+y)-\psi(x)] \subset \left\{(x,y)\in\mathbb{R}_*^2 : x\leq R,\ x+y\geq\overline{\delta}>0\right\}$ for suitable $\overline{\delta}>0$ and $R>0$. As a consequence, using (3.2) we deduce that

$$
\begin{aligned}
|I_1| &\leq c\int_{(0,\varepsilon R)}\int_{(\overline{\delta}\varepsilon/2,\varepsilon R)} y^{-\lambda}x^{1+\gamma+\lambda}\left|\psi\left(\frac{x+y}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}\right)\right|\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\
&\leq C(\psi)\frac{1}{\varepsilon}\int_{(0,\varepsilon R)} y^{1-\lambda}\Phi(\mathrm{d}y)\int_{(\overline{\delta}\varepsilon/2,\varepsilon R)} x^{1+\gamma+\lambda}\Phi(\mathrm{d}x)\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}
$$

Using again (3.2) we have

$$
\begin{aligned}
|I_2| &= \int_{\mathbb{R}_*}\int_{(0,y)} xK(x,y)\left|\psi\left(\frac{x+y}{\varepsilon}\right)-\psi\left(\frac{x}{\varepsilon}\right)\right|\Phi(\mathrm{d}x)\Phi(\mathrm{d}y)\\
&\leq C(\psi)\int_{(\varepsilon\delta/2,\infty)} y^{\gamma+\lambda}\Phi(\mathrm{d}y)\int_{(0,\varepsilon R)} x^{1-\lambda}\Phi(\mathrm{d}x)\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}
$$

Hence $\lim_{\varepsilon\to 0}\int_{\mathbb{R}_*} J_\phi(\varepsilon\xi)\varphi(\xi)\mathrm{d}\xi = 0$ and therefore $L=0$.

Step 4: Φ is zero near zero

Since $L=0$ we deduce by (6.7) that

$$
W_\phi(z) = \int_0^z \left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}-\xi\right)\phi(\xi)\mathrm{d}\xi \quad \text{for } z\in(0,\overline{\xi}). \tag{6.11}
$$

Or, equivalently, since $J_\phi\in BV(\mathbb{R}_*)$

$$
J_\phi(z)+\left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2z^2}{1-\gamma}\right)\phi(z) = \int_0^z\left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}-\xi\right)\phi(\xi)\mathrm{d}\xi \quad \text{for } a.e. z\in(0,\overline{\xi}). \tag{6.12}
$$

We now rewrite (6.12) in the following way

$$
\begin{aligned}
&\int_0^z\int_{z-x}^z xK(x,y)\Phi(\mathrm{d}y)\Phi(\mathrm{d}x)+\int_0^z\int_z^\infty xK(x,y)\Phi(\mathrm{d}y)\Phi(\mathrm{d}x)\\
&+\left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}}-\frac{2z^2}{1-\gamma}\right)\phi(z) = \int_0^z\left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}-\xi\right)\phi(\xi)\mathrm{d}\xi.
\end{aligned}
$$

If Φ is different to zero near the origin, then

$$
\begin{aligned}
\int_0^z\int_z^\infty xK(x,y)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y) &\geq c_1\int_0^z x^{1-\lambda}\Phi(\mathrm{d}x)\int_z^\infty y^{\gamma+\lambda}\Phi(\mathrm{d}y)\\
&\quad + c_1\int_0^z x^{1+\gamma+\lambda}\Phi(\mathrm{d}x)\int_z^\infty y^{-\lambda}\Phi(\mathrm{d}y)\\
&\geq C_+\int_0^z x^{1-\lambda}\Phi(\mathrm{d}x)
\end{aligned}
$$

where we have used that for sufficiently small z we have that

$$
\int_z^\infty y^{\gamma+\lambda}\Phi(\mathrm{d}y)\geq C_+
$$

for a strictly positive constant C_+.

This implies that for z small we have that

$$
\begin{aligned}
0 &= J_\phi(z) + \left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2z^2}{1-\gamma}\right)\phi(z) - \int_0^z \left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \xi\right)\phi(\xi)\mathrm{d}\xi \\
&\geq C_+ \int_0^z x^{1-\lambda}\phi(x)\mathrm{d}x + \left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2z^2}{1-\gamma}\right)\phi(z) \\
&\quad - \int_0^z \left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \xi\right)\phi(\xi)\mathrm{d}\xi \\
&\geq \left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2z^2}{1-\gamma}\right)\phi(z) + \int_0^z \left(C_+\xi^{1-\lambda} - \frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}\right)\phi(\xi)\mathrm{d}\xi.
\end{aligned}
$$

Since we are assuming that Φ is different from zero near the origin and since $\gamma+2\lambda > 1$, we deduce that $\int_0^z \left(C_+\xi^{1-\lambda} - \frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda}\right)\phi(\xi)\mathrm{d}\xi > 0$. This, together with the fact that for $z \leq \overline{\xi}$ we have $\left(\frac{z^{\gamma+\lambda+1}}{M_{\gamma+\lambda}} - \frac{2z^2}{1-\gamma}\right) \geq 0$ leads to a contradiction. Hence there exists a $\delta > 0$ such that $\phi(x) = 0$ for $x \in (0, \delta)$. We now prove that $\phi(x) = 0$ for every $x < \overline{\xi}$. To this end we define the function H as

$$
H(z) := \int_0^z \left(\frac{1}{M_{\gamma+\lambda}}\xi^{\gamma+\lambda} - \xi\right)\phi(\xi)\mathrm{d}\xi.
$$

Due to (6.12) we have that H satisfies

$$
\frac{d}{dz}H(z) = H(z)Q(z) - J_\phi(z)Q(z), \quad H(\delta) = 0 \quad \text{for } a.e.\ z \in (\delta, \overline{\xi}) \tag{6.13}
$$

where

$$
Q(z) := \frac{(1-\gamma)(z^{\gamma+\lambda} - M_{\gamma+\lambda}z)}{(1-\gamma)z^{\gamma+\lambda+1} - 2M_{\gamma+\lambda}z^2} > 0, \quad z \in (\delta, \overline{\xi}). \tag{6.14}
$$

The solution of Eq. (6.13) is the function

$$
H(z) = -\int_\delta^z \frac{J_\phi(s)}{Q(s)} e^{\int_s^z Q(x)\mathrm{d}x}\mathrm{d}s. \tag{6.15}
$$

Expression (6.15) for H implies that H is negative for $z \in (\delta, \overline{\xi})$. On the other hand, if we assume that ϕ is different from zero in $(0, \overline{\xi})$ we deduce that H is positive on $z \in (\delta, \overline{\xi})$, by definition. This contradiction implies that $H(z) = 0$ for $z \in (\delta, \overline{\xi})$. □

Remark 6.3. We remark that Eq. (6.11) is valid both for $\gamma + 2\lambda > 1$ and for $\gamma + 2\lambda = 1$, but the same argument we used to prove that $\Phi((0, \delta)) = 0$ does not work in the case $\gamma + 2\lambda = 1$.

Proof of Theorem 6.1. As in the proof of Theorem 5.1 we prove that if $\Phi((0, \delta)) = 0$ for some $\delta > 0$ then $\Phi((0, \rho(M_{\gamma+\lambda}))) = 0$ for $\rho(M_{\gamma+\lambda})$ given by (3.4). To conclude the proof combine this with Proposition 5.14 and Proposition 5.17 with Proposition 6.2. □

6.2. Non-existence results

Proof of Theorem 3.5. We consider now the case $\gamma + 2\lambda > 1$ and $\gamma \leq -1$ in which $\gamma + \lambda$ could be bigger than zero or smaller than zero, see Fig. 1. We will provide a proof that relies on the technical proof of Proposition 6.2. We proceed by contradiction. Assume that a self-similar solution exists. Then thanks to Proposition 6.2 we know that $\Phi((0, \delta)) = 0$ for some $\delta > 0$. Additionally, from Proposition 5.17 we know that $\lim_{x\to\infty} \Phi(x) = 0$ exponentially.

We can therefore consider the test function $\varphi \equiv 1$ in Eq. (3.3). This implies

$$0 \leq -\frac{1+\gamma}{1-\gamma} \int_{\mathbb{R}_*} \Phi(\mathrm{d}x) = -\frac{1}{2} \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K(x, y)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y) < 0,$$

which is a contradiction.

Assume now that $\gamma + 2\lambda = 1$ and that $\gamma \leq -1$. Notice that in this case we necessarily have that $\gamma + \lambda \leq 0$. Let us assume by contradiction that a self-similar solution exists and is such that

$$\int_{(0,1]} x^{-\lambda}\Phi(\mathrm{d}x) < \infty.$$

We show that

$$\int_{\mathbb{R}_*} \Phi(\mathrm{d}x) < \infty.$$

Notice that this is true by assumption when $\gamma + \lambda = 0$, therefore we restrict the attention to the case $\gamma + \lambda < 0$. To this end we adapt the argument used in the proof of Proposition 5.15, we refer there for the technical aspects, and prove that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda}\Phi(\mathrm{d}x) < \infty \quad \text{implies} \quad \int_{\mathbb{R}_*} x^{\gamma+\lambda+\overline{\delta}}\Phi(\mathrm{d}x) < \infty$$

when $0 < -\gamma - \lambda < \overline{\delta} < 1 - \gamma - \lambda$. To see this we consider $\psi(x) = x^{\gamma+\lambda+\overline{\delta}-1}$ in Eq. (5.48) to deduce that

$$\frac{2\overline{\delta}}{1-\gamma} M_{\gamma+\lambda+\overline{\delta}} \leq \frac{M_{2(\gamma+\lambda)+\overline{\delta}-1}}{M_{\gamma+\lambda}}.$$

Now we notice that our choice of $\overline{\delta}$ implies that

$$-\lambda \leq 2(\gamma + \lambda) + \overline{\delta} - 1 \leq \gamma + \lambda.$$

Hence $M_{2(\gamma+\lambda)+\overline{\delta}-1} < \infty$ and the desired conclusion follows.

Similarly, we can prove that

$$\int_{\mathbb{R}_*} x^{\gamma+\lambda+n\overline{\delta}}\Phi(\mathrm{d}x) < \infty \quad \text{implies} \quad \int_{\mathbb{R}_*} x^{\gamma+\lambda+(n+1)\overline{\delta}}\Phi(\mathrm{d}x) < \infty.$$

As a consequence, taking n sufficiently large, we deduce that $M_0(\Phi) < \infty$. We can therefore consider the test function $\varphi \equiv 1$ in Eq. (3.3). This implies

$$0 \leq -\frac{1+\gamma}{1-\gamma} \int_{\mathbb{R}_*} \Phi(\mathrm{d}x) = -\frac{1}{2} \int_{\mathbb{R}_*} \int_{\mathbb{R}_*} K(x, y)\Phi(\mathrm{d}x)\Phi(\mathrm{d}y) < 0,$$

which is a contradiction. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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