Applied General Topology
© Universidad Politécnica de Valencia
Volume 9, No. 1, 2008
pp. 15-19

Cancellation of 3-Point Topological Spaces

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Abstract. The cancellation problem, which goes back to S. Ulam [2], is formulated as follows:
Given topological spaces X, Y, Z, under what circumstances does $X \times Z \approx Y \times Z$ ($\approx$ meaning homeomorphic to) imply $X \approx Y$?
In [1] it is proved that, for T_0 topological spaces and denoting by S the Sierpinski space, if $X \times S \approx Y \times S$ then $X \approx Y$.
This note concerns all nine (up to homeomorphism) 3-point spaces, which are given in [4].

2000 AMS Classification: 54B10

Keywords: Homeomorphism, Cancellation problem, 3-point spaces.

1. Two cancellation results

Below X and Y denote T_1 topological spaces.

Proposition 1.1. *Let S be a topological space with a unique closed singleton $\{p\}$. If there is a homeomorphism $\phi : X\times S \to Y\times S$ then $\phi(X\times\{p\}) = Y\times\{p\}$.*

Proof. We shall show that $\phi(X\times\{p\}) \subset Y\times\{p\}$ which, using similar arguments, will be enough to prove that $\phi(X \times \{p\}) = Y \times \{p\}$ and, consequently, that $X \approx Y$.

Let us suppose that for some $x \in X, y \in Y$ and $q \in S \setminus \{p\}$ we have $\phi(x, p) = (y, q)$. Then $\{(y, q)\}$ is closed and, therefore, $(Y \times S) \setminus \{(y, q)\}$ is open.

Let r belong to the topological closure of $\{q\}, r \neq q$. Then $(y, r) \in (Y \times S) \setminus \{(y, q)\}$ and we must have open sets U_y, U_r, containing y and r, respectively, such that $U_y \times U_r \subset (Y \times S) \setminus \{(y, q)\}$. We reach a contradiction since (y, q) belongs to $U_y \times U_r$. □

*The second named author gratefully acknowledges financial support from *Fundação para a Ciência e Tecnologia*, Lisboa, Portugal.

An example of such an S is obtained as follows. Let S be a set with 4 elements at least. Let $a, b \in S$ and denote by S_1 the complement of the subset they form. Take then as basis for a topology on S the set $\{\{a\}, \{a,b\}, S_1\}$. If S happens to have just 4 points then it is the only minimal, universal space with such a number of elements [3].

Proposition 1.2. *Let S be a topological space with a dense, open singleton $\{p\}$ and such that, for every $q \in S \setminus \{p\}$, the topological closure of $\{q\}$ is finite. If there is a homeomorphism $\phi : X \times S \to Y \times S$ then $\phi(X \times \{p\}) = Y \times \{p\}$.*

Proof. Let $\{p\}$ be an open, dense singleton in S. We will show that $\phi(X \times \{p\}) = Y \times \{p\}$ which, as observed before, is enough to conclude that $X \approx Y$.

Assume that for some $x \in X, y \in Y$ and $q \neq p$ we have $\phi(x,p) = (y,q)$. Consider the closed set $\{y\} \times \overline{\{q\}}$, the bar denoting closure, its image $\phi^{-1}(\{y\} \times \overline{\{q\}})$, which is also closed, and suppose that $\overline{\{q\}}$ has s elements. Also, observe that $p \notin \overline{\{q\}}$.

Since (x,p) belongs to $\phi^{-1}(\{y\} \times \overline{\{q\}})$ and this set has s elements, there is an r in $\overline{\{q\}}$ such that (x,r) does not belong to this set. There are then open sets U_x, U_r, containing x and r, respectively, with $U_x \times U_r \subset (X \times S) \setminus \phi^{-1}(\{y\} \times \overline{\{q\}})$. We have a contradiction since $(x,p) \in U_x \times U_r$. □

An example for S can be the following *Door* space. Let S be a set and fix $p \in S$. Define $U \subset S$ to be open if it is empty or contains p.

2. 3-point spaces

We go on assuming that X, Y are T_1 topological spaces though such assumption is not used in Propositions 2.1 and 2.2 below.

If we now consider $S = \{a, b, c\}$ to be one of the 3-point spaces [4], we see that Propositions 1.1 and 1.2 of §1 allow us to deduce immediately that S can be cancelled except in the following cases

- S is discrete,
- S has $\{\{a\}, \{b\}, \{a,c\}\}$ as a topological basis,
- S is trivial.

If S is discrete the situation is not as simple as one might be led to think.

Let us take the following example. Let $S = Z$, here Z stands for the integers with the discrete topology, and consider the discrete spaces $X = \{0, 1, \ldots, n-1\}, n \geq 2, Y = \{0\}$. Now define $\phi : \{0, 1, \ldots, n-1\} \times Z \to \{0\} \times Z$ by $\phi(x,r) = (0, nr + x)$. This map is a homeomorphism and however Z cannot be cancelled.

We can say something when the spaces X, Y have a finite number of connected components.

Proposition 2.1. *Let S be a finite discrete space and assume that X has a finite number of connected components. If $X \times S \approx Y \times S$ then $X \approx Y$.*

Proof. The connected components of $X \times S$ or $Y \times S$ are of the type $X' \times \{x\}, Y' \times \{y\}$, where X', Y' are components of X and Y, respectively. It follows that Y has the same number of components as X.

Let us consider in the sets of connected components of X and connected components of Y the homeomorphism equivalence relation and take an equivalence class of components of X, say $\{X_1, \ldots, X_k\}$. The subspace $\bigcup_{i=1}^{k} X_i \times S$ has kn components, where n is the cardinal of S. The same happens with $\phi(\bigcup_{i=1}^{k} X_i \times S)$, where ϕ is a homeomorphism between $X \times S$ and $Y \times S$.

Let $p \in S$. For every $i = 1, \ldots, k$, $\phi(X_i \times \{p\}) = Y_i \times \{q_i\}$, where the q_i's belong to S and the Y_i's are components of Y homeomorphic to the X_i's. Assume that the equivalence class to which the Y_i's belong is $\{Y_1, \ldots, Y_l\}$. Then $\phi(\bigcup_{i=1}^{k} X_i \times \{p\}) \subset \bigcup_{j=1}^{l} Y_j \times S$. Consequently, also $\phi(\bigcup_{i=1}^{k} X_i \times S) \subset \bigcup_{j=1}^{l} Y_j \times S$.

Using the inverse homeomorphism ϕ^{-1}, we are led to conclude that the reverse inclusion holds and, therefore, $\phi(\bigcup_{i=1}^{k} X_i \times S) = \bigcup_{j=1}^{l} Y_j \times S$. So $\bigcup_{i=1}^{k} X_i \times S$ and $\bigcup_{j=1}^{l} Y_j \times S$ have the same number of components and it follows that $k = l$.

From each component class in X choose a representative and use ϕ to establish a homeomorphism between that representative and a component in Y. These homeomorphisms can then be used to conclude that every component of X is homeomorphic to a component of Y. Since components are closed and finite in number, X is homeomorphic to Y. □

Proposition 2.2. *Let X and Y be topological spaces with the same finite number of connected components and S be a discrete space. Assume, moreover, that neither space has two homeomorphic components. If $X \times S \approx Y \times S$ then $X \approx Y$.*

Proof. Let $X_i, i = 1, \ldots, n$, be the components of X and fix $p \in S$.

If ϕ is a homeomorphism between $X \times S$ and $Y \times S$ then there are $q_i \in S, i = 1, \ldots, n$, such that $\phi(X_i \times \{p\}) = Y_i \times \{q_i\}, i = 1, \ldots, n$, where, due to our assumption on the non-existence of homeomorphic components, the Y_i's are the components of Y. Hence ϕ induces a homeomorphism $\phi_i : X_i \to Y_i, i = 1, \ldots, n$.

Again, since the number of components is finite and they are closed, the ϕ_i's can be used to obtain a homeomorphism between X and Y. □

Proposition 2.3. *Let S have $\{\{a\}, \{b\}, \{a, c\}\}$ as basis. If $\phi : X \times S \to Y \times S$ is a homeomorphism then $\phi(X \times \{b\}) = Y \times \{b\}$.*

Proof. Let $\pi_S : Y \times S \to S$ denote the standard projection. The image $\pi_S(\phi(X \times \{b\}))$ is open and, therefore, it is either $\{b\}$ or contains a.

Assume that for some $x \in X, y \in Y$ we have $\phi(x, b) = (y, a)$. The subset $\{(x, b)\}$ is closed and, consequently, the same happens with $\{(y, a)\}$. Hence $(Y \times S) \setminus \{(y, a)\}$ is open and contains (y, c). We must then have an open neighbourhood U_y of y such that $U_y \times \{a, c\} \subset (Y \times S) \setminus \{(y, a)\}$. Again we have a contradiction and $\phi(X \times \{b\}) = Y \times \{b\}$. □

To conclude the proof that a non-discrete 3-point space can be cancelled it only remains to deal with the case where S is trivial.

Above we have an example of a homeomorphism $\phi : X \times S \to Y \times S$ which does take a slice $X \times \{x\}$ onto a slice $Y \times \{y\}$. More examples can be obtained.

Take $X = Y$, with at least 2 elements, a trivial space S with also, at least, 2 elements and let $\psi : S \to S$ be a fixed point free bijection. Fix $x_0 \in X$ and define $\phi : X \times S \to X \times S$ by $\phi(x, s) = (x, s)$, for $x \neq x_0$, and $\phi(x_0, s) = (x_0, \psi(s))$.

Then ϕ is a bijection and $\phi(\{x\} \times S) = \{x\} \times S$, for $x \in X$. Since open sets in $X \times S$ are of the form $U \times S$, U open in X, and $\phi(U \times S) = U \times S$, ϕ is a homeomorphism. Obviously no slice $X \times \{x\}$ is mapped onto a similar slice.

Proposition 2.4. *Let S be a finite trivial space. If $X \times S \approx Y \times S$ then $X \approx Y$.*

Proof. Open (closed) sets in $X \times S$ and $Y \times S$ are of the form $U \times S$, where U is open (closed).

We are going to define $f : X \to Y$ as follows. Let $x \in X$. Then $\{x\}$ is closed and so are $\{x\} \times S$ and $\phi(\{x\} \times S)$, where $\phi : X \times S \to Y \times S$ is a homeomorphism. Hence $\phi(\{x\} \times S) = C \times S$, for some closed set C in Y. Since S is finite, C is a singleton and we make $\{f(x)\} = C$.

This way we obtain an f which is a bijection since we began with a bijective ϕ.

If C is closed in X, $\phi(C \times S) = f(C) \times S$ is closed in $Y \times S$. Consequently $f(C)$ is closed in Y. Therefore f is closed and f^{-1} is continuous.

Taking ϕ^{-1}, we would conclude that f is continuous the same way. □

We can now state.

Theorem 2.5. *For X and Y T_1 topological spaces and S a non-discrete 3-point topological space, if $X \times S \approx Y \times S$ then $X \approx Y$.*

3. A particular case

We will no longer assume X, Y to be T_1 and will suppose that S has a unique isolated point a. Moreover, the singleton $\{a\}$ will be assumed to be closed. That is, for instance, the case where $S = \{a, b, c\}$ and $\{\{a\}, \{b, c\}\}$ is an open basis.

Proposition 3.1. *Let S have a unique isolated point a. Assume that $\{a\}$ is closed. For X, Y connected with, at least, an isolated point each, if $\phi : X \times S \to Y \times S$ is a homeomorphism then $\phi(X \times \{a\}) = Y \times \{a\}$.*

Proof. Let $\pi_S : Y \times S \to S$ denote the standard projection, as before.

The image $\pi_S(\phi(X \times \{a\}))$ is open and connected. Therefore it is either $\{a\}$ or some open, connected subset of S, which naturally does not contain a.

Let the latter be the case. If $x \in X$ is an isolated point then $\{(x, a)\}$ is open and the same happens to its image under $\pi_S \circ \phi$. This is impossible because $\{a\}$ is the unique open singleton of S. □

Examples of spaces satisfying the conditions of Proposition 3.1 are, again, some *Door* spaces.

Let Z be a set. Fix $p \in Z$ and define $U \subset Z$ to be open if $U = Z$ or $p \notin U$.

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Received July 2006

Accepted November 2006

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