

ON GIT QUOTIENTS OF HILBERT AND CHOW SCHEMES OF CURVES

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ABSTRACT. The aim of this note is to announce some results on the GIT problem for the Hilbert and Chow scheme of curves of degree d and genus g in the projective space of dimension $d - g$, whose full details will appear in [6]. In particular, we extend the previous results of L. Caporaso up to $d > 4(2g - 2)$ and we observe that this is sharp. In the range $2(2g - 2) < d < \frac{7}{2}(2g - 2)$, we get a complete new description of the GIT quotient. As a corollary, we get a new compactification of the universal Jacobian over the moduli space of pseudo-stable curves.

1. MOTIVATION

One of the first successful applications of Geometric Invariant Theory (GIT for short) was the construction of the moduli space M_g of smooth curves of genus $g \geq 2$ together with its compactification \overline{M}_g via stable curves, as carried out by Mumford ([20]) and Gieseker ([11]). Indeed, the moduli space of stable curves was constructed as a GIT quotient of the locally closed subset of a suitable Hilbert scheme (as in [11]) or Chow scheme (as in [20]) parametrizing n -canonically embedded curves, for $n \geq 5$ (see also [12, Chap. 4, Sec. C] or [18, Sec. 3]).

Recently, there has been a lot of interest in extending the above GIT analysis to smaller values of n , specially in connection with the so called Hassett-Keel program, whose ultimate goal is to find the minimal model of M_g via the successive constructions of modular birational models of \overline{M}_g (see [9] and [2] for nice overviews). The first work in this direction is due to Schubert, who described in [21] the GIT quotient of the locus of 3-canonically embedded curves (of genus $g \geq 3$) in the Chow scheme as the coarse moduli space $\overline{M}_g^{\text{ps}}$ of pseudo-stable curves (or

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p -stable curves for short), i.e., reduced, connected, projective curves with finite automorphism group, whose only singularities are nodes and ordinary cusps, and which have no elliptic tails. Indeed, it was shown by Hyeon-Morrison in [16] that one gets the same GIT quotient for the Hilbert scheme of 3-canonically embedded curves and for the Hilbert or Chow scheme of 4-canonically embedded curves. Later, Hassett-Hyeon constructed in [13] a modular map $T : \overline{M}_g \rightarrow \overline{M}_g^{\text{ps}}$ which on geometric points sends a stable curve onto the p -stable curve obtained by contracting all its elliptic tails to cusps. Moreover the authors of loc. cit. identified the map T with the first contraction in the Hassett-Keel program for \overline{M}_g . Finally, the GIT quotient of the Hilbert and Chow scheme of 2-canonically embedded curves was studied in great detail by Hassett-Hyeon in [14]. For some partial results on the GIT quotient for the Hilbert scheme of 1-canonically embedded curves, see the works of Alper-Fedorchuk-Smyth ([3], [4]) and of Fedorchuk-Jensen ([8]).

From the point of view of constructing new projective birational models of \overline{M}_g , it is of course natural to restrict the GIT analysis to the locally closed subset inside the Hilbert or the Chow scheme parametrizing n -canonical embedded curves. However, the problem of describing the whole GIT quotient seems very natural and interesting too. The first result in this direction is the breakthrough work of Lucia Caporaso [7], where she describes the GIT quotient of the Hilbert scheme of connected curves of genus $g \geq 3$ and degree $d \geq 10(2g - 2)$ in \mathbb{P}^{d-g} . The GIT quotient that she obtains is indeed a modular compactification $\overline{J}_{d,g}$ of the universal Jacobian $J_{d,g}$, which is the moduli scheme parametrizing pairs (C, L) where C is a smooth curve of genus g and L is a line bundle on C of degree d . Note that recently Li-Wang in [17] have given a different proof of the Caporaso's result for $d \gg 0$.

Our work is motivated by the following:

Problem(I): *Describe the GIT quotient of the Hilbert and Chow scheme of curves of genus g and degree d in \mathbb{P}^{d-g} , as d decreases with respect to g .*

Ideally, one would like then to interpret the different GIT quotients obtained (as d decreases with respect to g) as first steps in a suitable ‘‘Hassett-Keel’’ program for Caporaso’s compactified universal Jacobian $\overline{J}_{d,g}$ (see also Question B and the discussion following it). We hope to come back to this project in a future work.

In order to describe our results, we need to introduce some notation.

2. SETUP

We work over an algebraically closed field k (of arbitrary characteristic). For an integer $g \geq 2$ and any natural number d , denote by Hilb_d the Hilbert scheme of curves of degree d and arithmetic genus g in $\mathbb{P}^{d-g} = \mathbb{P}(V)$; denote by Chow_d the Chow scheme of 1-cycles of degree d in \mathbb{P}^{d-g} and by

$$\text{Ch} : \text{Hilb}_d \rightarrow \text{Chow}_d$$

the surjective map sending a one dimensional subscheme $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ to its 1-cycle. The linear algebraic group $\text{SL}(V) \cong \text{SL}_{d-g+1}$ acts naturally on Hilb_d and Chow_d in such a way that Ch is an equivariant map. The action of $\text{SL}(V)$ on Hilb_d and Chow_d can be naturally linearized as follows.

For any $m \gg 0$, setting $P(m) := md + 1 - g$, the Hilbert scheme Hilb_d admits a $\text{SL}(V)$ -equivariant embedding

$$(1) \quad j_m : \text{Hilb}_d \hookrightarrow \text{Gr}(P(m), \text{Sym}^m V^\vee) \hookrightarrow \mathbb{P} \left(\bigwedge^{P(m)} \text{Sym}^m V^\vee \right) := \mathbb{P},$$

where $\text{Gr}(P(m), \text{Sym}^m V^\vee)$ is the Grassmannian variety parametrizing $P(m)$ -dimensional quotients of $\text{Sym}^m V^\vee$, which embeds in $\mathbb{P} \left(\bigwedge^{P(m)} \text{Sym}^m V^\vee \right)$ via the Plücker embedding. Explicitly, the map j_m sends $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ into

$$\begin{aligned} j_m([X \subset \mathbb{P}^{d-g}]) &:= [\text{Sym}^m V^\vee \rightarrow H^0(X, \mathcal{O}_X(m))] \\ &\in \text{Gr}(P(m), \text{Sym}^m V^\vee) \hookrightarrow \mathbb{P} \left(\bigwedge^{P(m)} \text{Sym}^m V^\vee \right). \end{aligned}$$

We refer to [19, Lect. 15] for details. From the embedding (1), we get an ample $\text{SL}(V)$ -linearized line bundle $\Lambda_m := j_m^* \mathcal{O}_{\mathbb{P}}(1)$ (for $m \gg 0$) and we denote by

$$\text{Hilb}_d^{s,m} \subseteq \text{Hilb}_d^{ps,m} \subseteq \text{Hilb}_d^{ss,m} \subseteq \text{Hilb}_d$$

the locus of points that are, respectively, stable, polystable and semistable with respect to Λ_m . It is well-known (see [14, Sec. 3.6]) that $\text{Hilb}_d^{s,m}$, $\text{Hilb}_d^{ps,m}$ and $\text{Hilb}_d^{ss,m}$ are constant for $m \gg 0$, and we set

$$\begin{cases} \text{Hilb}_d^s := \text{Hilb}_d^{s,m} & \text{for } m \gg 0, \\ \text{Hilb}_d^{ps} := \text{Hilb}_d^{ps,m} & \text{for } m \gg 0, \\ \text{Hilb}_d^{ss} := \text{Hilb}_d^{ss,m} & \text{for } m \gg 0. \end{cases}$$

If $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d^s$ (resp. Hilb_d^{ps} , resp. Hilb_d^{ss}), we say that $[X \subset \mathbb{P}^{d-g}]$ is *Hilbert stable* (resp. *Hilbert polystable*, resp. *Hilbert semistable*).

The Chow scheme Chow_d admits a $\text{SL}(V)$ -equivariant embedding

$$(2) \quad \text{Chow}_d \xrightarrow{i} \mathbb{P}(\otimes^2 \text{Sym}^d V^\vee) := \mathbb{P}'$$

obtained by sending a 1-cycle Z of degree d in \mathbb{P}^{d-g} into the hyperplane of $\otimes^2 \text{Sym}^d V^\vee$ generated by all the elements $F \otimes G \in \text{Sym}^d V^\vee$ such that $Z \cap \{F = 0\} \cap \{G = 0\} \neq \emptyset$. We refer to [19, Lec. 16] for details. From the embedding (2), we get an ample $\text{SL}(V)$ -linearized line bundle $\Lambda := i^* \mathcal{O}_{\mathbb{P}'}(1)$ and we denote by

$$\text{Chow}_d^s \subseteq \text{Chow}_d^{ps} \subseteq \text{Chow}_d^{ss} \subseteq \text{Chow}_d$$

the locus of points of Chow_d that are, respectively, stable, polystable or semistable with respect to Λ . We say that $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ is *Chow semistable* (resp. *Chow polystable*, resp. *Chow stable*) if its image $\text{Ch}([X \subset \mathbb{P}^{d-g}])$ belongs to Chow_d^{ss} (resp. Chow_d^{ps} , resp. Chow_d^s). The relation between Hilbert (semi)stability and Chow (semi)stability is given by the following chain of open inclusions (see [14, Prop. 3.13])

$$(3) \quad \text{Ch}^{-1}(\text{Chow}_d^s) \subseteq \text{Hilb}_d^s \subseteq \text{Hilb}_d^{ss} \subseteq \text{Ch}^{-1}(\text{Chow}_d^{ss}) \subseteq \text{Hilb}_d.$$

In particular, there is a natural morphism of GIT-quotients:

$$\text{Hilb}_d^{ss} // \text{SL}(V) \rightarrow \text{Chow}_d^{ss} // \text{SL}(V).$$

Note that, in general, there is no obvious relation between Hilb_d^{ps} and $\text{Ch}^{-1}(\text{Chow}_d^{ps})$: for example, according to [14, Prop. 11.6 and Prop. 11.8], there are 2-canonical curves that are Hilbert polystable but not Chow polystable and conversely.

We can now reformulate Problem(I) in the following form.

Problem(II): *Describe the points $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ that are Hilbert or Chow (semi, poly) stable, as d decreases with respect to g .*

The aim of this note is to announce some partial results on the above Problem(II). Full details will appear in [6].

3. RESULTS

Our partial answer to the above Problem(II) will require some conditions on the singularities of X and some conditions on the multidegree of the line bundle $\mathcal{O}_X(1)$. Let us introduce the relevant definitions.

Definition 1.

- (i) A curve X is said to be *quasi-stable* if it is obtained from a stable curve Y by “blowing up” some of its nodes, i.e. by taking the partial normalization of Y at some of its nodes and inserting a \mathbb{P}^1 connecting the two branches of each node.
- (ii) A curve X is said to be *quasi-p-stable* if it is obtained from a p-stable curve Y by “blowing up” some of its nodes (as before) and “blowing up” some of its cusps, i.e. by taking the partial normalization of Y at some of its cusps and inserting a \mathbb{P}^1 tangent to the branch point of each cusp.

Given a quasi-stable or a quasi-p-stable curve X , we call the \mathbb{P}^1 's inserted by blowing up nodes or cusps of Y the *exceptional components*, and we denote by $X_{\text{exc}} \subset X$ the union of all of them.

Definition 2. Let X be a quasi-stable or a quasi-p-stable curve of genus $g \geq 2$ and let L be a line bundle on X of degree d . We say that:

- (i) L is *balanced* if for each subcurve $Z \subset X$ the following inequality (called the basic inequality) is satisfied

$$(*) \quad \left| \deg_Z L - \frac{d}{2g-2} \deg_Z(\omega_X) \right| \leq \frac{|Z \cap Z^c|}{2},$$

where $|Z \cap Z^c|$ denotes the length of the 0-dimensional subscheme of X obtained as the scheme-theoretic intersection of Z with the complementary subcurve $Z^c := \overline{X \setminus Z}$.

- (ii) L is *properly balanced* if L is balanced and the degree of L on each exceptional component of X is 1.
- (iii) L is *strictly balanced* if L is properly balanced and the basic inequality (*) is strict except possibly for the subcurves Z such that $Z \cap Z^c \subset X_{\text{exc}}$.
- (iv) L is *stably balanced* if L is properly balanced and the basic inequality (*) is strict except possibly for the subcurves Z such that Z or Z^c is entirely contained in X_{exc} .

The inequality (*) first appeared in the work of Mumford [20, Prop. 5.5] and Gieseker [11, Prop. 1.0.11]. See also [7, Sec. 3.1], where (*) is called the “Basic Inequality”.

We are now ready to state the main results of [6]. Our first result deals with high values of the degree d .

Theorem A. *Consider a point $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ with $d > 4(2g - 2)$ and $g \geq 2$; assume moreover that X is connected. Then the following conditions are equivalent:*

- (i) $[X \subset \mathbb{P}^{d-g}]$ is Hilbert semistable (resp. polystable, resp. stable);
- (ii) $[X \subset \mathbb{P}^{d-g}]$ is Chow semistable (resp. polystable, resp. stable);
- (iii) X is quasi-stable and $\mathcal{O}_X(1)$ is balanced (resp. strictly balanced, resp. stably balanced).

In each of the above cases, $X \subset \mathbb{P}^{d-g}$ is non-degenerate and linearly normal, and $\mathcal{O}_X(1)$ is non-special.

Moreover, the Hilbert or Chow GIT quotient is geometric (i.e. all the Hilbert or Chow semistable points are stable) if and only if $\gcd(2g - 2, d - g + 1) = 1$.

The above Theorem was proved by Caporaso in [7] for $d \geq 10(2g - 2)$ and for Hilbert (semi-, poly-)stability. We remark that the hypothesis $d > 4(2g - 2)$ in the above Theorem A is sharp: in [16] it is proved that a 4-canonical p-stable curve (which in particular can have cusps) is Hilbert stable while a 4-canonical stable curve with an elliptic tail is not Hilbert semistable.

We then investigate what happens if $d \leq 4(2g - 2)$ and we get a complete answer in the case $2(2g - 2) < d < \frac{7}{2}(2g - 2)$ and $g \geq 3$.

Theorem B. *Consider a point $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ with $2(2g - 2) < d < \frac{7}{2}(2g - 2)$ and $g \geq 3$; assume moreover that X is connected. Then the following conditions are equivalent:*

- (i) $[X \subset \mathbb{P}^{d-g}]$ is Hilbert semistable (resp. polystable, resp. stable);
- (ii) $[X \subset \mathbb{P}^{d-g}]$ is Chow semistable (resp. polystable, resp. stable);
- (iii) X is quasi-p-stable and $\mathcal{O}_X(1)$ is balanced (resp. strictly balanced, resp. stably balanced).

In each of the above cases, $X \subset \mathbb{P}^{d-g}$ is non-degenerate and linearly normal, and $\mathcal{O}_X(1)$ is non-special.

Moreover, the Hilbert or Chow GIT quotient is geometric (i.e. all the Hilbert or Chow semistable points are stable) if and only if $\gcd(2g - 2, d - g + 1) = 1$.

We note that the conditions on the degree d and the genus g in the above Theorem B are sharp. Indeed, if $d = 2(2g - 2)$ then it follows from [14, Thm. 2.14] that there are 2-canonical Hilbert stable curves having arbitrary tacnodes and not only tacnodes obtained by blowing up a cusp as in Definition 1(ii). On the other hand, if $d = \frac{7}{2}(2g - 2)$ (resp. $d > \frac{7}{2}(2g - 2)$) then it follows from [11, Prop. 1.0.6, Case 2] that a point $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ such that X is a quasi-p-stable but not p-stable curve is not Chow stable (resp. Chow semistable)¹. Finally, if $g = 3$ then Hyeon-Lee proved in [15] that a 3-canonical irreducible p-stable curve with one cusp is not Hilbert polystable.

As an application of Theorem B, we get a new compactification of the universal Jacobian $J_{d,g}$ over the moduli space of p-stable curves of genus g . To this aim, consider the category fibered in groupoids $\overline{\mathcal{J}}_{d,g}^{\text{ps}}$ over the category of schemes, whose fiber over a scheme S is the groupoid of families of quasi-p-stable curves over S endowed with a line bundle whose restriction to the geometric fibers is properly balanced.

¹We thank Fabio Felici for pointing out to us the relevance of [11, Prop. 1.0.6, Case 2].

Theorem C. *Let $g \geq 3$ and $d \in \mathbb{Z}$.*

- (1) $\overline{\mathcal{J}}_{d,g}^{\text{ps}}$ is a smooth, irreducible Artin stack of finite type over k and of dimension $4g - 4$. Moreover $\overline{\mathcal{J}}_{d,g}^{\text{ps}}$ is universally closed and weakly separated (in the sense of [5]).
- (2) $\overline{\mathcal{J}}_{d,g}^{\text{ps}}$ admits an adequate moduli space $\overline{J}_{d,g}^{\text{ps}}$ (in the sense of [1]), which is a normal irreducible projective variety of dimension $4g - 3$ containing $J_{d,g}$ as an open subvariety. Moreover, if $\text{char}(k) = 0$, then $\overline{J}_{d,g}^{\text{ps}}$ has rational singularities, hence it is Cohen-Macaulay.
- (3) There exists a commutative digram

$$\begin{array}{ccc} \overline{\mathcal{J}}_{d,g}^{\text{ps}} & \longrightarrow & \overline{J}_{d,g}^{\text{ps}} \\ \Psi^{\text{ps}} \downarrow & & \downarrow \Phi^{\text{ps}} \\ \overline{\mathcal{M}}_g^{\text{ps}} & \longrightarrow & \overline{M}_g^{\text{ps}} \end{array}$$

where Ψ^{ps} is surjective, universally closed and weakly separated (in the sense of [5]) and Φ^{ps} is surjective and projective with equidimensional fibers of dimension g .

- (4) If $\text{char}(k) = 0$ or $\text{char}(k) = p > 0$ is bigger than the order of the automorphism group of any p -stable curve of genus g , then for any $X \in \overline{M}_g^{\text{ps}}$, the fiber $(\Phi^{\text{ps}})^{-1}(X)$ is isomorphic to $\overline{\text{Jac}}_d(X)/\text{Aut}(X)$, where $\overline{\text{Jac}}_d(X)$ is the Simpson's compactified Jacobian of X parametrizing S -equivalence classes of rank 1, torsion-free sheaves on X that are slope-semistable with respect to ω_X .
- (5) If $2(2g - 2) < d < \frac{7}{2}(2g - 2)$ then $\overline{\mathcal{J}}_{d,g}^{\text{ps}} \cong [H_d/GL(r + 1)]$ and $\overline{J}_{d,g}^{\text{ps}} \cong H_d/GL(r + 1)$, where $H_d \subset \text{Hilb}_d$ is the open subset consisting of points $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ such that X is connected and $[X \subset \mathbb{P}^{d-g}]$ is Hilbert semistable (or equivalently, Chow semistable).

The proof of the above Theorems will appear in [6].

4. OPEN QUESTIONS

The above results leave unsolved some natural questions for further investigation, which we discuss briefly here.

The first question is, of course, the following

Question A. *Describe the Hilbert and Chow (semi-,poly-)stable points of Hilb_d in the case where $\frac{7}{2}(2g - 2) \leq d \leq 4(2g - 2)$.*

Indeed, as observed before, Theorems A and B do not hold for these values of d . In [6, Thm. 5.1], we proved the following partial result: if $[X \subset \mathbb{P}^{d-g}]$ is Chow semistable with X connected then $\mathcal{O}_X(1)$ is balanced and X is a reduced curve whose singularities are at most nodes, cusps and tacnodes at which two components of X meet, one of which is a line of \mathbb{P}^{d-g} . Further progress has been made by Fabio Felici in [10].

By analogy with the contraction map $T : \overline{M}_g^{\text{ps}} \rightarrow \overline{M}_g$ constructed by Hassett-Hyeon in [13], the following problem arises very naturally.

Question B. Construct a map $\tilde{T} : \bar{J}_{d,g} \rightarrow \bar{J}_{d,g}^{\text{ps}}$ fitting into the following commutative diagram

$$\begin{array}{ccc} \bar{J}_{d,g} & \xrightarrow{\tilde{T}} & \bar{J}_{d,g}^{\text{ps}} \\ \Phi^s \downarrow & & \downarrow \Phi^{ps} \\ \bar{M}_g & \xrightarrow{T} & \bar{M}_g^{\text{ps}}, \end{array}$$

where $\bar{J}_{d,g} \xrightarrow{\Phi^s} \bar{M}_g$ is Caporaso's compactification of the universal Jacobian over \bar{M}_g .

More generally, one would like to set up a ‘‘Hassett-Keel’’ program for $\bar{J}_{d,g}$ and give an interpretation of the above map \tilde{T} as the first step in this program.

Finally, the reader may have noticed that in Theorems A and B we have characterized points $[X \subset \mathbb{P}^{d-g}] \in \text{Hilb}_d$ that are Hilbert or Chow (semi,poly)stable under the assumption that X is connected. Indeed, it is easy to prove that, at least for $d > 2(2g - 2)$, the condition of being connected is both open and closed inside the Hilbert or Chow semistable locus. However, we do not know an answer to the following

Question C. Are there connected components inside the Hilbert or Chow GIT semistable locus made entirely of non-connected curves?

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