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Letter

A note on the Lipkin model in arbitrary fermion number

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A possible form of the Lipkin model obeying the $su(6)$ algebra is presented. It is a natural generalization from the idea for the $su(4)$ algebra recently proposed by the present authors. All the relations appearing in the present form can be expressed in terms of spherical tensors in the $su(2)$ algebras. For specifying the linearly independent basis completely, twenty parameters are introduced. It is concluded that, in these parameters, five quantum numbers determine the minimum weight state. The other five parameters in twenty parameters are also regarded as quantum numbers.

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This paper is a continuation of two papers, recently published by the present authors [1,2]. Hereafter, these two will be referred to as (I) and (II), respectively. In these papers, we treated the Lipkin model [3,4] with arbitrary single-particle levels and fermion number. In (I), an idea of how to construct the minimum weight state, which is the starting point of the algebraic approach, was proposed. In (II), we discussed how to express the linearly independent basis built on a chosen minimum weight state. The present paper aims mainly at supplementing the results of (II) with a discussion on the $su(6)$ algebra, which we promised in (II).

First, let us consider the Lipkin model obeying the $su(n)$ algebra for the case with $n = 2m$ in a rather general framework. Here, m denotes an integer. Since the total fermion number N is a constant of motion, we omit the discussion on N. Our present argument is restricted to the case with even integer n with $n = 4, 6, \ldots$, i.e., $n = 2m$ with $m = 2, 3, \ldots$. If the generators in the $su(n)$ algebra are expressed appropriately, we can show that this model includes m $su(2)$ subalgebras. This point has been shown in (II) and the generators are given in the relation (II.2.3). Therefore, as the total sum, we can define the $su(2)$ algebra ($\widetilde{S}_{\pm,0}$) that will play a central role in our approach. Of course, the generators in each $su(2)$ algebra form a vector. Further, in (II), we showed that the $su(n)$ Lipkin model includes one $su(m)$ subalgebra, all generators in which are scalar for ($\widetilde{S}_{\pm,0}$) and the rest

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form $m(m-1)/2$ vectors. They are given in the relations (II.2.9) and (II.2.10)–(II.2.12), respectively. Generally, the minimum weight states in the $su(n)$ algebra are specified by $(n-1)$ quantum numbers. In the present case, we can decompose the number $(n-1)$ into two parts:

$$n-1 = m + (m-1) \qquad (n = 2m). \tag{1}$$

As has already been mentioned, the case with $n = 2m$ includes m $su(2)$ algebras and one $su(m)$ algebra. The first and second terms, m and $(m-1)$, represent, respectively, the numbers of the quantum numbers related to m $su(2)$ algebras and of one $su(m)$ algebra for the minimum weight state.

Next, let us consider the orthogonal set constructed by operating “certain operators" on any minimum weight state. We will call them the excited-state generating operators. They should be expressed in terms of $n(n-1)/2$ quantum numbers coming from the relation

$$\frac{1}{2}\left((n^2-1)-(n-1)\right) = \frac{1}{2}n(n-1) = m(2m-1) \qquad (n = 2m). \tag{2}$$

The number $n(n-1)/2$ is equal to that of the $su(n)$ generators with the types $\widetilde{S}^p(n)$ ($p = 1, 2, \ldots, n-1$) and $\widetilde{S}^p_q(n)$ ($p > q = 1, 2, \ldots, n-2$): $(n-1)+(n-1)(n-2)/2 = n(n-1)/2$. We will call them the raising operators and their Hermitian conjugates the lowering operators. The definitions of $\widetilde{S}^p(n)$ and $\widetilde{S}^p_q(n)$ have been given in the relation (I.2.2). We know that there exist m $su(2)$ subalgebras and one $su(m)$ algebra and then, in the excited-state generating operators, the m and $m(m-1)/2$ operators are related to these subalgebras, respectively. Therefore, we must investigate the rest, the number of which is given by

$$m(2m-1) - \left(m + \frac{1}{2}m(m-1)\right) = 3 \cdot \frac{1}{2}m(m-1). \tag{3}$$

It should be noted that the number $m(m-1)/2$ is just equal to that of the vector operators for $(\widetilde{S}_{\pm,0})$, which are presented in (II). With the use of these three types of operators, we can expect to obtain a possible idea for constructing the excited-state generating operators. Further, we notice that the relation (3) can be decomposed into

$$3 \cdot \frac{1}{2}m(m-1) = \frac{1}{2}m(m-1) + 2 \cdot \frac{1}{2}m(m-1). \tag{4}$$

The first term corresponds to the number of the raising or lowering operators in the $su(m)$ subalgebra. It is well known that we can construct a tensor operator with two parameters in terms of a vector, e.g., the solid harmonics $\mathcal{Y}_{l,l_0} = r^l Y_{ll_0}(\theta\phi)$ is constructed in terms of the position vector ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$). In our case, there exist $m(m-1)/2$ vectors and each gives us a tensor operator in the $su(2)$ subalgebras, which is specified by two parameters. This argument leads us to the following: The second term represents the total number of parameters contained in the operators that are built by $m(m-1)/2$ vectors. On the other hand, the first term $m(m-1)/2$ corresponds to the number of lowering operators in the $su(m)$ subalgebra. One merit of our idea may be as follows: All generators of the $su(n)$ Lipkin model are expressed in terms of the spherical tensors for $(\widetilde{S}_{\pm,0})$ and then we can apply the technique of the angular momentum coupling rule.

As a simple example of the Lipkin model, we will give a brief summary of the case with $n = 4$, i.e., $m = 2$, which has been discussed in (II), but the form in this paper is slightly different from that of (II). The generators in the present two $su(2)$ subalgebras are copied from the relation (II.5.9):

$$\widetilde{S}_+(1) = \widetilde{S}_2^3\ , \qquad \widetilde{S}_-(1) = \widetilde{S}_3^2\ , \qquad \widetilde{S}_0(1) = \frac{1}{2}\left(\widetilde{S}_3^3 - \widetilde{S}_2^2\right)\ , \tag{4a}$$

$$\widetilde{S}_+(2) = \widetilde{S}^1\ , \qquad \widetilde{S}_-(2) = \widetilde{S}_1\ , \qquad \widetilde{S}_0(2) = \frac{1}{2}\widetilde{S}_1^1. \tag{4b}$$

The sum is given by

$$\widetilde{S}_{\pm,0} = \widetilde{S}_{\pm,0}(1) + \widetilde{S}_{\pm,0}(2). \tag{5}$$

The generators in the $su(m = 2)$ subalgebra are expressed as

$$\widetilde{R}_+ = \widetilde{S}_1^3 + \widetilde{S}^2\ , \qquad \widetilde{R}_- = \widetilde{S}_3^1 + \widetilde{S}_2\ , \qquad \widetilde{R}_0 = \frac{1}{2}\left(\widetilde{S}_3^3 + \widetilde{S}_2^2 - \widetilde{S}_1^1\right). \tag{6}$$

Since $\widetilde{R}_{\pm,0}$ are scalars for $(\widetilde{S}_{\pm,0})$, we have the relation

$$\left[\ \text{any of } \widetilde{S}_{\pm,0}\ ,\ \text{any of } \widetilde{R}_{\pm,0}\ \right] = 0. \tag{7}$$

The vector operators for $(\widetilde{S}_{\pm,0})$ are given as

$$\widetilde{R}^{1,+1} = -\widetilde{S}^3\ , \qquad \widetilde{R}^{1,0} = \frac{1}{\sqrt{2}}\left(\widetilde{S}_1^3 - \widetilde{S}^2\right)\ , \qquad \widetilde{R}^{1,-1} = \widetilde{S}_1^2\ , \tag{8a}$$

$$\widetilde{R}_{1,+1} = -\widetilde{S}_3\ , \qquad \widetilde{R}_{1,0} = \frac{1}{\sqrt{2}}\left(\widetilde{S}_3^1 - \widetilde{S}_2\right)\ , \qquad \widetilde{R}_{1,-1} = \widetilde{S}_2^1\ , \tag{8b}$$

The relations (6)–(8) are copied from the relations (II.5.11)–(II.5.14). The vectors $\widetilde{R}_{\pm,0}$ satisfy

$$\left[\ \widetilde{R}_+\ ,\ \widetilde{R}^{1,\nu}\ \right] = 0\ , \qquad \left[\ \widetilde{R}_0\ ,\ \widetilde{R}^{1,\nu}\ \right] = \widetilde{R}^{1,\nu} \qquad (\nu = \pm,\ 0). \tag{9}$$

The minimum weight state is expressed as $|\rho, \sigma^1, \sigma^2\rangle$, where the eigenvalues of $\widetilde{R}_0$, $\widetilde{S}_0(1)$, and $\widetilde{S}_0(2)$ are denoted as $-\rho$, $-\sigma^1$, and $-\sigma^2$, respectively. The explicit form of $|\rho, \sigma^1, \sigma^2\rangle$ is given in (I). Then, the eigenstate of $\left(\widetilde{\boldsymbol{R}}^2, \widetilde{R}_0\right)$, $\left(\widetilde{\boldsymbol{S}}(1)^2, \widetilde{S}_0(1)\right)$, and $\left(\widetilde{\boldsymbol{S}}(2)^2, \widetilde{S}_0(2)\right)$ with the eigenvalues $(\rho(\rho+1), \rho_0)$, $(\sigma^1(\sigma^1+1), \sigma_0^1)$, and $(\sigma^2(\sigma^2+1), \sigma_0^2)$ can be expressed as

$$|\rho_0, \sigma_0^1, \sigma_0^2; \rho, \sigma^1, \sigma^2\rangle = f_{\rho\rho_0} f_{\sigma^1\sigma_0^1} f_{\sigma^2\sigma_0^2} \widetilde{P}^{\rho\rho_0,\sigma^1\sigma_0^1,\sigma^2\sigma_0^2} |\rho, \sigma^1, \sigma^2\rangle\ , \tag{10a}$$

$$\widetilde{P}^{\rho\rho_0,\sigma^1\sigma_0^1,\sigma^2\sigma_0^2} = \left(\widetilde{R}_+\right)^{\rho+\rho_0} \left(\widetilde{S}_+(1)\right)^{\sigma^1+\sigma_0^1} \left(\widetilde{S}_+(2)\right)^{\sigma^2+\sigma_0^2}\ , \tag{10b}$$

$$f_{\tau\tau_0} = \sqrt{\frac{(\tau-\tau_0)!}{(2\tau)!(\tau+\tau_0)!}} \qquad (\tau = \rho,\ \sigma^1,\ \sigma^2). \tag{11}$$

The eigenstate of $(\widetilde{\boldsymbol{S}}^2, \widetilde{S}_0)$ with the eigenvalue $(\sigma(\sigma+1), \sigma_0)$ is given by

$$|\rho\rho_0, \sigma^1, \sigma^2, \sigma\sigma_0\rangle = \sum_{\sigma_0^1\sigma_0^2} \langle\sigma^1\sigma_0^1\sigma^2\sigma_0^2|\sigma\sigma_0\rangle |\rho_0, \sigma_0^1, \sigma_0^2; \rho, \sigma^1, \sigma^2\rangle. \tag{12}$$

We are investigating the case with $n = 4$, i.e., $m = 2$. Therefore, as can be seen in the relation (3), we must search for the operator characterized by three quantum numbers for obtaining the excited-state generating operators. For this task, in (II), the following form is adopted:

$$\widetilde{R}^{l,l_0;\text{ł},\lambda_0} = \left(\overrightarrow{R}_-\right)^{l-\lambda_0} \left(\overrightarrow{S}_-\right)^{l-l_0} \left(-\widetilde{S}^3\right)^l. \tag{13}$$

Here, we adopt the notation for $\widetilde{O}$ and $\widetilde{A}$ in the form

$$\left(\overrightarrow{O}\right)^n \widetilde{A} = \underbrace{[\widetilde{O}, \ldots, [\widetilde{O}, [\widetilde{O}}_{n}, \widetilde{A}]] \cdots]. \tag{14}$$

Since $\overrightarrow{S}_+ \left(-\widetilde{S}^3\right)^l = \overrightarrow{R}_+ \left(-\widetilde{S}^3\right)^l = 0$ and $\overrightarrow{S}_0 \left(-\widetilde{S}^3\right)^l = \overrightarrow{R}_0 \left(-\widetilde{S}^3\right)^l = l \left(-\widetilde{S}^3\right)^l$, $\widetilde{R}^{l,l_0;l,\lambda_0}$ can be regarded as a spherical tensor operator for the $su(2)$ algebras $(\widetilde{S}_{\pm,0})$ and $(\widetilde{R}_{\pm,0})$ with rank l. Operating $\widetilde{R}^{l,l_0;l,\lambda_0}$ on the state (12) and applying the angular momentum coupling rule, we have

$$|l, rr_0, ss_0, \sigma; \rho\sigma^1\sigma^2\rangle = \sum_{\lambda_0 l_0} \langle l\lambda_0 \rho\rho_0 | rr_0\rangle \langle ll_0 \sigma\sigma_0 | ss_0\rangle \widetilde{R}^{l,l_0;l,\lambda_0} |\rho\rho_0, \sigma^1, \sigma^2, \sigma\sigma_0\rangle. \tag{15}$$

However, the set composed of the states (15) cannot be regarded as orthogonal. The reasons for this are as follows: (1) the symbol l is not a quantum number, but a parameter, the value of which is given from the outside, and (2) the operator $\widetilde{R}^{l,l_0;l,\lambda_0}$ contains degrees of freedom that are not contained in the states $|\rho\rho_0, \sigma^1, \sigma^2, \sigma\sigma_0\rangle$, such as $-\widetilde{S}^3$. Therefore, they form a linearly independent basis and by an appropriate method, e.g., the Schmidt method, we must construct the orthogonal set. In the sense mentioned above, the excited-state generating operator is given by

$$\begin{aligned}\widetilde{R}^{l,l_0;l,\lambda_0} \cdot \widetilde{P}^{\rho\rho_0,\sigma^1\sigma^1_0,\sigma^2\sigma^2_0} &= \left(\overrightarrow{R}_-\right)^{l-\lambda_0} \left(\overrightarrow{S}_-\right)^{l-l_0} \left(-\widetilde{S}^3\right)^l \\ &\times \left(\widetilde{R}_+\right)^{\rho+\rho_0} \left(\widetilde{S}_+(1)\right)^{\sigma^1+\sigma^1_0} \left(\widetilde{S}_+(2)\right)^{\sigma^2+\sigma^2_0}. \end{aligned} \tag{16}$$

The above is the summary of (II) for the case with $n = 4$, i.e., $m = 2$. In preparation for the discussion on the case with $n = 6$, i.e., $m = 3$, we rewrite the relation (13) in a form slightly different from the original. The form $(\overrightarrow{S}_-)^{l-l_0}(-\widetilde{S}^3)^l$ can be rewritten as

$$\left(\overrightarrow{S}_-\right)^{l-l_0} \left(-\widetilde{S}^3\right)^l = \left(\overrightarrow{S}_-\right)^{l-l_0} \left(\widetilde{R}^{1,+1}\right)^l = \widetilde{Z}^{l,l_0}. \tag{17}$$

Here, $\widetilde{Z}^{l,l_0}$ is of the form

$$\begin{aligned}\widetilde{Z}^{l,l_0} &= \sqrt{\frac{l!}{(2l-1)!!}} \sum_{\lambda} \left(\frac{1}{\sqrt{2}}\right)^{l-\lambda} \frac{\sqrt{(l+l_0)!(l-l_0)!}}{\left(\frac{l+l_0-\lambda}{2}\right)!\lambda!\left(\frac{l-l_0-\lambda}{2}\right)!} \\ &\times \left(\widetilde{R}^{1,+1}\right)^{\frac{l+l_0-\lambda}{2}} \left(\widetilde{R}^{1,0}\right)^{\lambda} \left(\widetilde{R}^{1,-1}\right)^{\frac{l-l_0-\lambda}{2}}. \end{aligned} \tag{18}$$

The sum for λ obeys the following condition: if $|l - l_0| =$ even or odd, λ cannot be odd or even, respectively. The operator $\widetilde{Z}^{l,l_0}$ is a tensor with rank l for $(\widetilde{S}_{\pm,0})$:

$$\begin{aligned}\overrightarrow{S}_\pm \widetilde{Z}^{l,l_0} &= \sqrt{(l \mp l_0)(l \pm l_0 + 1)}\widetilde{Z}^{l,l_0 \pm 1}, \\ \overrightarrow{S}_0 \widetilde{Z}^{l,l_0} &= l_0 \widetilde{Z}^{l,l_0}. \end{aligned} \tag{19}$$

In the case that $(\widetilde{R}^{1,\pm 1}, \widetilde{R}^{1,0})$ is a position vector $(r_{\pm 1} = \mp(x \pm iy)/\sqrt{2}, r_0 = z)$, $\widetilde{Z}^{l,l_0}$ is reduced to the solid harmonics:

$$\widetilde{Z}^{l,l_0} \longrightarrow \sqrt{\frac{4\pi l!}{(2l+1)!!}} \mathcal{Y}_{l,l_0} \qquad \left(\mathcal{Y}_{l,l_0} = r^l Y_{ll_0}(\theta\phi)\right). \tag{20}$$

An important property of $\widetilde{Z}^{l,l_0}$ is as follows:

$$\overrightarrow{R}_+\widetilde{Z}^{l,l_0} = 0\,, \qquad \overrightarrow{R}_0\widetilde{Z}^{l,l_0} = l\widetilde{Z}^{l,l_0}. \tag{21}$$

With the use of the relation (9), we are able to obtain the property (21).

Under the above preparation, we will investigate the case with $n = 6$, i.e., $m = 3$. This case includes three $su(2)$ subalgebras in the form

$$\widetilde{S}_+(1) = \widetilde{S}_4^5\,, \qquad \widetilde{S}_-(1) = \widetilde{S}_5^4\,, \qquad \widetilde{S}_0(1) = \frac{1}{2}\left(\widetilde{S}_5^5 - \widetilde{S}_4^4\right)\,, \tag{22a}$$

$$\widetilde{S}_+(2) = \widetilde{S}_2^3\,, \qquad \widetilde{S}_-(2) = \widetilde{S}_3^2\,, \qquad \widetilde{S}_0(2) = \frac{1}{2}\left(\widetilde{S}_3^3 - \widetilde{S}_2^2\right)\,, \tag{22b}$$

$$\widetilde{S}_+(3) = \widetilde{S}^1\,, \qquad \widetilde{S}_-(3) = \widetilde{S}_1\,, \qquad \widetilde{S}_0(3) = \frac{1}{2}\widetilde{S}_1^1. \tag{22c}$$

The total sum of the above is denoted as

$$\widetilde{S}_{\pm,0} = \widetilde{S}_{\pm,0}(1) + \widetilde{S}_{\pm,0}(2) + \widetilde{S}_{\pm,0}(3). \tag{23}$$

The eight generators in the $su(m = 3)$ subalgebra, which are scalars for $(\widetilde{S}_{\pm,0})$, are written down as follows:

$$\widetilde{R}_+ = \widetilde{S}_3^5 + \widetilde{S}_2^4\,, \qquad \widetilde{R}_- = \widetilde{S}_5^3 + \widetilde{S}_4^2\,, \qquad \widetilde{R}_0 = \frac{1}{2}\left(\widetilde{S}_5^5 + \widetilde{S}_4^4 - \widetilde{S}_3^3 - \widetilde{S}_2^2\right)\,, \tag{24a}$$

$$\widetilde{R}^{\frac{1}{2},\frac{1}{2}} = \widetilde{S}_1^5 + \widetilde{S}^4\,, \qquad \widetilde{R}_{\frac{1}{2},\frac{1}{2}} = \widetilde{S}_5^1 + \widetilde{S}_4\,,$$

$$\widetilde{R}^{\frac{1}{2},-\frac{1}{2}} = \widetilde{S}_1^3 + \widetilde{S}^2\,, \qquad \widetilde{R}_{\frac{1}{2},-\frac{1}{2}} = \widetilde{S}_3^1 + \widetilde{S}_2\,, \tag{24b}$$

$$\widetilde{R} = \frac{1}{2}\left(\widetilde{S}_5^5 + \widetilde{S}_4^4 + \widetilde{S}_3^3 + \widetilde{S}_2^2\right) - \widetilde{S}_1^1. \tag{24c}$$

The set (24a) forms the $su(2)$ algebra and $(\widetilde{R}^{\frac{1}{2},\frac{1}{2}}, \widetilde{R}^{\frac{1}{2},-\frac{1}{2}})$ and $(-\widetilde{R}_{\frac{1}{2},-\frac{1}{2}}, \widetilde{R}_{\frac{1}{2},\frac{1}{2}})$ are spinors for $(\widetilde{R}_{\pm,0})$. The present case includes three vectors for $(\widetilde{S}_{\pm,0})$:

$$\widetilde{R}^{1,+1}(1) = -\widetilde{S}^5\,, \qquad \widetilde{R}^{1,0}(1) = \frac{1}{\sqrt{2}}\left(\widetilde{S}_1^5 - \widetilde{S}^4\right)\,, \qquad \widetilde{R}^{1,-1}(1) = \widetilde{S}_1^4\,, \tag{25a}$$

$$\widetilde{R}^{1,+1}(2) = -\widetilde{S}_2^5\,, \qquad \widetilde{R}^{1,0}(2) = \frac{1}{\sqrt{2}}\left(\widetilde{S}_3^5 - \widetilde{S}_2^4\right)\,, \qquad \widetilde{R}^{1,-1}(2) = \widetilde{S}_3^4\,, \tag{25b}$$

$$\widetilde{R}^{1,+1}(3) = -\widetilde{S}^3\,, \qquad \widetilde{R}^{1,0}(3) = \frac{1}{\sqrt{2}}\left(\widetilde{S}_1^3 - \widetilde{S}^2\right)\,, \qquad \widetilde{R}^{1,-1}(3) = \widetilde{S}_1^2. \tag{25c}$$

The Hermitian conjugates of the vectors are omitted. The expressions (22), (24), and (25) are obtained by putting $m = 3$ in the relations (II.2.3) and (II.2.9)–(II.2.12). For $\nu = \pm 1,\ 0$, the vector $(\widetilde{R}^{1,\nu}(1))$ satisfies the relation

$$\overrightarrow{R}_+\widetilde{R}^{1,\nu}(1) = \overrightarrow{R}^{\frac{1}{2},\frac{1}{2}}\widetilde{R}^{1,\nu}(1) = \overrightarrow{R}^{\frac{1}{2},-\frac{1}{2}}\widetilde{R}^{1,\nu}(1) = 0\,, \tag{26a}$$

$$\overrightarrow{R}_0\widetilde{R}^{1,\nu}(1) = \frac{1}{2}\widetilde{R}^{1,\nu}(1)\,, \qquad \overrightarrow{R}\widetilde{R}^{1,\nu}(1) = \frac{3}{2}\widetilde{R}^{1,\nu}(1). \tag{26b}$$

For $\nu = \pm 1,\ 0$ and $k = 2,\ 3$, two vectors $(\widetilde{R}^{1,\nu}(k))$ obey the relation

$$\overrightarrow{R}_{+}\widetilde{R}^{1,\nu}(k) = \delta_{k,3}\widetilde{R}^{1,\nu}(k)\ , \quad \overrightarrow{R}^{\frac{1}{2},\frac{1}{2}}\widetilde{R}^{1,\nu}(k) = 0\ , \quad \overrightarrow{R}^{\frac{1}{2},-\frac{1}{2}}\widetilde{R}^{1,\nu}(k) = -\delta_{k,2}\widetilde{R}^{1,\nu}(k)\ , \tag{27a}$$

$$\overrightarrow{R}_{0}\widetilde{R}^{1,\nu}(k) = \left(\delta_{k,2} - \delta_{k,3}\cdot\frac{1}{2}\right)\widetilde{R}^{1,\nu}(k)\ , \quad \overrightarrow{R}\widetilde{R}^{1,\nu}(k) = \delta_{k,3}\cdot\frac{3}{2}\widetilde{R}^{1,\nu}(k). \tag{27b}$$

The above are the relations for our present case ($n = 6$). However, $\widetilde{R}^{1,\nu}(2)$ and $\widetilde{R}^{1,\nu'}(3)$ do not commute mutually, but they do commute with $\widetilde{R}^{1,\nu''}(1)$. We will treat the case with $n = 6$ as a natural generalization from the case with $n = 4$.

Now, in parallel with the $su(4)$ Lipkin model, we are able to give our scheme for obtaining the linearly independent basis for the $su(6)$ Lipkin model. In order to avoid unnecessary complications, we will not apply the angular momentum coupling rule, together with the associating numerical factors, such as the form (11). The minimum weight state is expressed in the form $|\rho,\rho^1,\sigma^1,\sigma^2,\sigma^3\rangle$, where $\rho,\rho^1,\sigma^1,\sigma^2$, and σ^3 denote the eigenvalues of $-\widetilde{R}$, $-\widetilde{R}_0$, $-\widetilde{S}_0(1)$, $-\widetilde{S}_0(2)$, and $-\widetilde{S}_0(3)$ defined in the relations (24) and (22), respectively. As a possible extension of $\widetilde{P}^{\rho\rho_0,\sigma^1\sigma^1_0,\sigma^2\sigma^2_0}$, shown in the relation (10b), we introduce the following operator:

$$\widetilde{P}^{\mu\mu_0,\rho^1\rho^1_0,\sigma^1\sigma^1_0,\sigma^2\sigma^2_0,\sigma^3\sigma^3_0} = \left(\widetilde{R}^{\frac{1}{2},\frac{1}{2}}\right)^{\mu+\mu_0}\left(\widetilde{R}^{\frac{1}{2},-\frac{1}{2}}\right)^{\mu-\mu_0}\left(\widetilde{R}_{+}\right)^{\rho^1+\rho^1_0}\left(\widetilde{S}_{+}(1)\right)^{\sigma^1+\sigma^1_0}\left(\widetilde{S}_{+}(2)\right)^{\sigma^2+\sigma^2_0}\left(\widetilde{S}_{+}(3)\right)^{\sigma^3+\sigma^3_0}. \tag{28}$$

With the use of the operator (28), we define the state

$$|\mu\mu_0,\rho^1_0,\sigma^1_0,\sigma^2_0,\sigma^3_0;\rho,\rho^1,\sigma^1,\sigma^2,\sigma^3\rangle = \widetilde{P}^{\mu\mu_0,\rho^1\rho^1_0,\sigma^1\sigma^1_0,\sigma^2\sigma^2_0,\sigma^3\sigma^3_0}|\rho,\rho^1,\sigma^1,\sigma^2,\sigma^3\rangle. \tag{29}$$

The state (29) corresponds to the state (10a) and is expressed in terms of eleven parameters, some of which play the role of quantum numbers.

The above is the first step in our approach to the case with $n = 6$. The linearly independent basis of our present case should be expressed in terms of twenty parameters in total. Therefore, we must further investigate how to consider nine parameters that may be related to three vectors. For this task, we extend the idea adopted in the case with $n = 4$. This case starts in the relation (9). The relation (9) for the vector (17) with (18) for the definition of $\widetilde{Z}^{l,l_0}$ leads us to the relation (21) for the raising operator $\widetilde{R}_{+}$ and the Hermitian operator $\widetilde{R}_0$. These two correspond to $(\widetilde{R}_{+},\widetilde{R}^{\frac{1}{2},\frac{1}{2}},\widetilde{R}^{\frac{1}{2},-\frac{1}{2}})$ and $(\widetilde{R}_0,\widetilde{R})$, respectively, for the present case. As can be seen in the relation (27a), the operation of $\overrightarrow{R}_{+}$ and $\overrightarrow{R}^{\frac{1}{2},\pm\frac{1}{2}}$ on the vectors labeled $k = 2,\ 3$ does not vanish. Thus, if we try to use the form extended from the relation (9), we must introduce new vectors. The following are possible candidates for these:

$$\mathring{R}^{1,\nu}(1) = \widetilde{R}^{1,\nu}(1) \quad (\nu = \pm 1,\ 0) \tag{30a}$$

$$\mathring{R}^{1,1}(k) = \frac{1}{\sqrt{2}}\left(\widetilde{R}^{1,1}(1)\widetilde{R}^{1,0}(k) - \widetilde{R}^{1,1}(k)\widetilde{R}^{1,0}(1)\right)\ ,$$

$$\mathring{R}^{1,0}(k) = \frac{1}{\sqrt{2}}\left(\widetilde{R}^{1,1}(1)\widetilde{R}^{1,-1}(k) - \widetilde{R}^{1,1}(k)\widetilde{R}^{1,-1}(1)\right)\ ,$$

$$\mathring{R}^{1,-1}(k) = \frac{1}{\sqrt{2}}\left(\widetilde{R}^{1,0}(1)\widetilde{R}^{1,-1}(k) - \widetilde{R}^{1,0}(k)\widetilde{R}^{1,-1}(1)\right). \tag{30b}$$

The vector (30b) is derived by the formula $Z_\nu = \sum_{\lambda,\mu}\langle 1\lambda 1\mu|1\nu\rangle X_\lambda Y_\mu$ for two vectors X and Y. The vectors (30) satisfy the following relation for $\nu = \pm 1,\ 0$ and $k = 1,\ 2,\ 3$:

$$\overrightarrow{R}_+\mathring{R}^{1,\nu}(k) = \overrightarrow{R}^{\frac{1}{2},\pm\frac{1}{2}}\mathring{R}^{1,\nu}(k) = 0\,, \tag{31a}$$

$$\overrightarrow{R}_0\mathring{R}^{1,\nu}(k) = \frac{1}{2}\left(1+\delta_{k,2}-\delta_{k,3}\right)\mathring{R}^{1,\nu}(k)\,,$$

$$\overrightarrow{R}\mathring{R}^{1,\nu}(k) = \frac{3}{2}\left(1+\delta_{k,3}\right)\mathring{R}^{1,\nu}(k). \tag{31b}$$

With the use of the vectors (31), we can define the operator

$$\mathring{Z}^{l^1l_0^1,l^2l_0^2,l^3l_0^3} = \widetilde{Z}^{l^1l_0^1}(1)\widetilde{Z}^{l^2l_0^2}(2)\widetilde{Z}^{l^3l_0^3}(3). \tag{32}$$

Here, $\widetilde{Z}^{l^kl_0^k}(k)$ $(k=1,\ 2,\ 3)$ is obtained by replacing ll_0 and $\widetilde{R}^{1,\nu}$ in the relation (18) with $l^kl_0^k$ and $\mathring{R}^{1,\nu}(k)$. As has already been mentioned, some pairs of $(\widetilde{R}^{1,\nu}(k))$ do not commute and, therefore, the ordering of $\widetilde{Z}^{l^kl_0^k}(k)$ in the definition (32) should be fixed beforehand, as is shown, e.g., in the definition (32). The operator (32) satisfies

$$\overrightarrow{R}_+\mathring{Z}^{l^1l_0^1,l^2l_0^2,l^3l_0^3} = \overrightarrow{R}^{\frac{1}{2},\pm\frac{1}{2}}\mathring{Z}^{l^1l_0^1,l^2l_0^2,l^3l_0^3} = 0\,, \tag{33a}$$

$$\overrightarrow{R}_0\mathring{Z}^{l^1l_0^1,l^2l_0^2,l^3l_0^3} = \frac{1}{2}\left(l^1+3l^2\right)\mathring{Z}^{l^1l_0^1,l^2l_0^2,l^3l_0^3}\,,$$

$$\overrightarrow{R}\mathring{Z}^{l^1l_0^1,l^2l_0^2,l^3l_0^3} = \frac{3}{2}\left(l^1+l^2+2l^3\right)\mathring{Z}^{l^1l_0^1,l^2l_0^2,l^3l_0^3}. \tag{33b}$$

We can see that the above is a natural extension of the relation (21).

The operator (32) is expressed in terms of six parameters (l^1, l_0^1), (l^2, l_0^2), and (l^3, l_0^3). Thus, in order to accomplish our task, we must search for a further three parameters. As can be seen in the relation (13), the case with $n = 4$ is completed by taking into account the lowering operator $\widetilde{R}_-$ in the form $\overrightarrow{R}_-$. The relation (33a) tells us that the operation of the raising operators $\widetilde{R}_+$ and $\widetilde{R}^{\frac{1}{2},\pm\frac{1}{2}}$ makes the results vanish. It may be enough to consider three lowering operators $\overrightarrow{R}_-$ and $\mp\overrightarrow{R}_{\frac{1}{2},\pm\frac{1}{2}}$ on the operator (32). First, we note that the operator (32) is nothing but a tensor specified by

$$l = l_0 = \frac{1}{2}(l^1+3l^2). \tag{34}$$

Then, the tensor operator specified by l and l_0 $(l_0 = -l,\ -l+1,\ldots,\ l-1,\ l)$ can be given in the form

$$\mathring{Z}_{ll_0}{}^{(l^1l_0^1,l^2l_0^2,l^3l_0^3)} = \left(\overrightarrow{R}_-\right)^{l-l_0}\mathring{Z}^{l^1l_0^1,l^2l_0^2,l^3l_0^3}. \tag{35}$$

Next, we consider the operators $\mp\widetilde{R}_{\frac{1}{2},\mp\frac{1}{2}}$, which form the spinor for $(\widetilde{R}_{\pm,0})$. As is well known, the tensor operator specified by λ and λ_0 $(\lambda_0 = -\lambda,\ -\lambda+1,\ldots,\ \lambda-1,\ \lambda)$ is constructed in the form

$$\widetilde{Y}_{\lambda\lambda_0} = \left(-\tilde{R}_{\frac{1}{2},-\frac{1}{2}}\right)^{\lambda+\lambda_0}\left(\widetilde{R}_{\frac{1}{2},\frac{1}{2}}\right)^{\lambda-\lambda_0}. \tag{36}$$

In the case of applying the angular momentum couping rule, it may be convenient to attach the numerical factor $g_{\lambda\lambda_0}$:

$$g_{\lambda\lambda_0} = \sqrt{\frac{(2\lambda)!}{(\lambda+\lambda_0)!(\lambda-\lambda_0)!}}. \tag{37}$$

Then, we introduce the operator

$$\overrightarrow{Y}_{\lambda\lambda_0} = \left(-\overrightarrow{R}_{\frac{1}{2},-\frac{1}{2}}\right)^{\lambda+\lambda_0}\left(\overrightarrow{R}_{\frac{1}{2},\frac{1}{2}}\right)^{\lambda-\lambda_0} . \tag{38}$$

The product of the operators (35) and (38) gives us an operator with nine parameters:

$$\begin{aligned}\widetilde{R}_{\lambda\lambda_0,ll_0}{}^{(l^1l_0^1,l^2l_0^2,l^3l_0^3)} &= \overrightarrow{Y}_{\lambda\lambda_0}\mathring{Z}_{ll_0}{}^{(l^1l_0^1,l^2l_0^2,l^3l_0^3)} \\ &= \left(-\overrightarrow{R}_{\frac{1}{2}-\frac{1}{2}}\right)^{\lambda+\lambda_0}\left(\overrightarrow{R}_{\frac{1}{2},\frac{1}{2}}\right)^{\lambda-\lambda_0}\left(\overrightarrow{R}_{-}\right)^{l-l_0}\mathring{Z}^{l^1l_0^1,l^2l_0^2,l^3l_0^3}.\end{aligned} \tag{39}$$

Thus, multiplying the operator (39) by (28), we obtain a state generating operator with fifteen parameters.

Up to now, we have not been concerned about the spherical tensor representation of our model with respect to the $su(2)$ algebras ($\widetilde{R}_{\pm,0}$) and ($\widetilde{S}_{\pm,0}$). Its explicit expression is omitted, but we will discuss the basic idea for this problem. The linearly independent basis obtained in this paper is expressed in terms of twenty parameters, of which at least five are related to the quantum numbers for the minimum weight state, i.e., ρ, ρ^1, σ^1, σ^2, and σ^3. Thus, it may be important to investigate which parameters play the role of quantum numbers. This is our final problem in this paper.

Application of the angular momentum coupling rule to the state (29) leads us to the following:

$$\mu\mu_0, \rho^1\rho_0^1 \quad\longrightarrow\quad \mu\rho^1; \eta\eta_0 , \tag{40a}$$

$$\sigma^1\sigma_0^1, \sigma^2\sigma_0^2, \sigma^3\sigma_0^3 \quad\longrightarrow\quad \sigma^1\sigma^2(\sigma^{12})\sigma^3; \sigma^{123}\sigma_0^{123} , \tag{40b}$$

$$|\mu-\rho^1| \le \eta \le \mu+\rho^1, \tag{41a}$$

$$\begin{aligned}&|\sigma^1-\sigma^2| \le \sigma^{12} \le \sigma^1+\sigma^2 , \\ &|\sigma^{12}-\sigma^3| \le \sigma^{123} \le \sigma^{12}+\sigma^3.\end{aligned} \tag{41b}$$

The inequalities (41a) and (41b) are related to the coupling rule for ($\widetilde{R}_{\pm,0}$) and ($\widetilde{S}_{\pm,0}$), respectively. If we note that ρ^1, σ^1, σ^2, and σ^3 denote the quantum numbers coming from the Casimir operator, μ, η, σ^{12}, and σ^{123} play the role of parameters obeying the inequality (41). Next, we consider the operator (38). In this case, the coupling rule gives us

$$\lambda\lambda_0, ll_0 \quad\longrightarrow\quad \lambda l; \xi\xi_0 , \tag{42a}$$

$$l^1l_0^1, l^2l_0^2, l^3l_0^3 \quad\longrightarrow\quad l^1l^2(l^{12})l^3; l^{123}l_0^{123} , \tag{42b}$$

$$|\lambda-l| \le \xi \le \lambda+l, \tag{43a}$$

$$\begin{aligned}&|l^1-l^2| \le l^{12} \le l^1+l^2 , \\ &|l^{12}-l^3| \le l^{123} \le l^{12}+l^3.\end{aligned} \tag{43b}$$

Under the relation (34), the seven parameters λ, ξ, l^1, l^2, l^{12}, l^3, and l^{123} have to be regarded as parameters obeying the inequality (43). Finally, we consider $\widetilde{\boldsymbol{R}}^2$, $\widetilde{R}_0$, $\widetilde{\boldsymbol{S}}^2$, and $\widetilde{S}_0$ and, further, $\widetilde{R}$. Let the eigenvalues of $\widetilde{\boldsymbol{R}}^2$, $\widetilde{R}_0$, $\widetilde{\boldsymbol{S}}^2$, and $\widetilde{S}_0$ denote $r(r+1)$, r_0, $s(s+1)$, and s_0, respectively. For ($\widetilde{\boldsymbol{R}}^2$, $\widetilde{R}_0$) and ($\widetilde{\boldsymbol{S}}^2$, $\widetilde{S}_0$), we have the relations

$$\xi\xi_0, \eta\eta_0 \quad\longrightarrow\quad \xi\eta; rr_0 , \tag{44a}$$

$$l^{123}l_0^{123}, \sigma^{123}\sigma_0^{123} \quad\longrightarrow\quad l^{123}\sigma^{123}; ss_0 , \tag{44b}$$

$$|\xi - \eta| \leq r \leq \xi + \eta \,, \tag{45a}$$

$$|l^{123} - \sigma^{123}| \leq s \leq l^{123} + \sigma^{123}. \tag{45b}$$

At the present stage, we can summarize our discussion as follows: The eight symbols ρ^1, σ^1, σ^2, σ^3, (r, r_0), and (s, s_0) denote the quantum numbers and the eleven λ, μ, ξ, η, σ^{12}, σ^{123}, l^1, l^2, l^{12}, l^3, and l^{123} must be regarded as the parameters, making nineteen in total. However, in the above analysis, there is one point missing. We must take account of ρ and R (the eigenvalue of $\widetilde{R}$). After straightforward calculation, we can show that our linearly independent basis is a set of the eigenstates of $\widetilde{R}$ and R can be expressed in the form

$$R = 3(\mu - \lambda) + \frac{3}{2}(l^1 + l^2 + 2l^3) - \rho. \tag{46}$$

Then, we have to add R and ρ to the quantum numbers. Therefore, one of the three parameters μ, λ, and $(l^1 + l^2 + 2l^3)/2$, e.g., μ, depends on the others if R and ρ are given.

Thus, we have a summary: In our linearly independent basis, ρ, ρ^1, σ^1, σ^2, σ^3, r, r_0, s, s_0, and R denote the quantum numbers and σ^{12}, σ^{123}, l^1, l^2, l^{12}, l^3, l^{123}, ξ, and η play the role of parameters. Then, with the use of an appropriate method, e.g., the Schmidt method, we can construct the orthogonal set for the $su(6)$ Lipkin model with arbitrary fermion numbers.

Including (I) and (II), this note has been devoted to a discussion of the Lipkin model, which belongs to the classical model in nuclear many-body theories. The authors have tried to stress that there still exist some problems that remain unsolved. In particular, it may be interesting to investigate the phase structure and/or phase transition in the $su(6)$ Lipkin model, such as those appearing in the usual $su(2)$ Lipkin model. This will be an interesting problem for the future.

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