This is the authors' version of the manuscript accepted to be published in Annals of Operations Research. The final version is available at: https://doi.org/10.1007/s10479-022-04903-0

A stochastic method for exploiting outranking relations in multicriteria choice problems

Luis C. Dias,[1] Humberto Rocha[2]

[1] University of Coimbra, CeBER, Faculty of Economics, Av Dias da Silva 165, 3004-512 Coimbra, Portugal. Email: lmcdias@fe.uc.pt (corresponding author)

[2] University of Coimbra, CeBER, Faculty of Economics, Av Dias da Silva 165, 3004-512 Coimbra, Portugal. Email: hrocha@fe.uc.pt

Abstract

The multicriteria decision aiding field offers many methods to support decision makers in comparing a list of alternatives. Among these, outranking methods such as ELECTRE are appreciated for avoiding full compensation among criteria, but outranking relations are difficult to exploit due to incompleteness and lack of transitivity. This work focuses on choice problems, proposing a stochastic exploitation method to select the most preferred alternative. It builds on the concept of Markov solution, which has become popular to select a winner in tournaments and voting problems. The proposed method can be used to exploit crisp outranking relations, valued outranking relations, or stochastic outranking relations. This can be a valuable addition to the toolbox for exploiting outranking relations as this work shows that solutions can be computed without much effort and guarantee some essential properties.

Keywords: Multiple criteria decision analysis, Outranking relations, ELECTRE, Choice problematic, Markov

1. Introduction

The field of Multicriteria Decision Aiding (MCDA) offers a wide variety of methods to support Decision Makers (DMs) who wish to evaluate or compare a list of alternatives (Belton & Stewart, 2002; Greco et al., 2016; Ishizaka & Nemery, 2013). These methods include outranking methods, which build and exploit binary relations on the set of alternatives. Such relations, called outranking relations, can be crisp (outranks / not outranks) or valued (an outranking degree in [0,1]), according to the method.

Typically, the use of an outranking method begins with a construction step, in which the crisp or valued outranking relation on a set of alternatives $A=\{a_1,\ldots,a_n\}$ is derived from data on the alternatives and from a number of parameters required by the method, namely those reflecting the preferences of the DMs. This is followed by the exploitation step, in which the outranking relation is analyzed in view of producing a recommendation, in terms of the choice, ranking, or sorting results. Outranking methods include the well-known ELECTRE (Dias & Mousseau, 2018; Figueira et al., 2005; B. Roy, 1991) and PROMETHEE (Behzadian et al., 2010; Brans et al., 1986), among other (Martel & Matarazzo, 2005). Outranking methods are particularly suited to DMs who wish to avoid complete compensation (substitution) among criteria, i.e., not allowing a very poor performance on one criterion to be compensated by a very good performance on some other criterion.

Outranking methods such as ELECTRE build and exploit crisp or valued outranking relations among pairs of alternatives. In ELECTRE, a statement a_i outranks a_j, denoted (a_iSa_j) means that a_i is at least as good as a_j (Figueira et al., 2005; Roy, 1991). When the focus is on relative or comparative problems, outranking relations are not easy to exploit because the outranking relation is not complete (some pairs of alternatives may have no relation) and not transitive (for instance cycles can occur). Methods to exploit such relations have been proposed, for the purposes of choice and especially for ranking problems. The latter include mainly two groups of methods. One group adapts concepts of social choice and voting methods, such as the Net Flow rule (Bouyssou, 1992), the Min rule (Pirlot, 1995), or prudent orders and their extensions (Dias & Lamboray, 2010). Another group follows optimization strategies seeking to obtain a good compatibility between the outranking relations and the obtained ranking (Fernandez et al., 2008; Fernandez & Leyva, 2004; Leyva López et al., 2021). For choice problems specifically, a well-known approach is to select a kernel of an outranking relation, as proposed by Bernard Roy (1968, 1996).

When the problem is stated as a choice problem statement, DMs wish to select the most preferred alternative (one alternative only) among a given set. For instance, if A={a_1, a_2, a_3} and S is such that a_1Sa_2 and a_1Sa_3 while $\neg(a_2Sa_1)$ and $\neg(a_3Sa_1)$, then a_1 should be chosen and, differently from a ranking problem, one does not care to differentiate between a_2 and a_3. The classical way of exploiting it is to define a subset of A, as small as possible, containing a shortlist of the candidates to be the best option. Roy (1968) suggested the use of the kernel concept from graph theory:

$$K \text{ is a kernel} \Leftrightarrow \begin{cases} \forall a_j \notin K, \exists a_i \in K: a_iSa_j \\ \forall a_i, a_j \in K, \neg(a_iSa_j) \end{cases}$$

Depending on how rich the S relation is, the kernel can contain a single winner, or several (sometimes many) potential winners. However, if the outranking relation contains cycles, a kernel method might not exist, or more than one kernel might exist, causing the need to break cycles by removing their weakest link or to consider cycles as a single alternative (an indifference class) (Roy & Bouyssou, 1993).

The above paragraph is just an example of the difficulties faced when exploiting a relation that is not transitive and not complete when it comes to selecting the most preferred alternative. Such problems also appear in the literature of social choice, in which the binary relation reflects the number of persons preferring one option to another, as well as in the literature on tournaments, in which the binary relation reflects the results of the confrontations among pairs of alternatives. To cope with these difficulties, several authors have independently proposed under different names the use of what we name in this work the Markov solution (Brandt et al., 2016), described in the following section.

This work contributes to the literature on exploiting outranking relations in MCDA by proposing and studying a stochastic method based on the concept of Markov solution (Brandt et al., 2016). To the best of our knowledge, no prior work has studied the concept of Markov solution in this context. For probabilistic outranking relations obtained, for instance, from a Stochastic Multicriteria Acceptability Analysis (SMAA) approach (Govindan et al., 2019; Hokkanen et al., 1998), the Markov solution can be readily defined, according to different variants proposed and studied in this work. For traditional (non-probabilistic) crisp or valued outranking relations, this work also proposes different variants for transforming them into a stochastic relation. Two desirable characteristics of the exploitation method for our specific context (monotonicity and dominance with regards to the evaluations of the alternatives on multiple criteria) are also demonstrated, considering the specificities of a probabilistic outranking function.

Following this introduction, Section 2 briefly reviews the concept of a Markov solution. Section 3 presents a method for a stochastic exploitation of a crisp or valued outranking relation. The method is illustrated on several examples in Section 4 and 5. Section 6 discusses some mathematical properties of this exploitation approach, and Section 7 concludes this article.

2. Background on the Markov solution

Let $A=\{a_1,\ldots,a_n\}$ denote a set of n alternatives the DM wishes to choose from. The rationale for the Markov exploitation strategy is the following. Suppose the DM is contemplating an alternative a_i and considering it as a provisional choice. When comparing it with other alternatives, the DM might wish to exchange alternative a_i for another one. Let p_{ij} denote the probability that the DM exchanges a_i for alternative a_j (j≠i) and let p_{ii} denote the probability that the DM keeps a_i. Naturally, one has

$$p_{ij} \geq 0 \text{ and } \sum_{j=1}^{n} p_{ij} = 1, \forall i$$

The Markov solution builds on ideas dating back to the work of Daniels (1969) and Moon & Pullman (1970). It consists in selecting the alternative with the highest probability of being chosen according to this process. It has been studied mainly in the context of determining a winner from the results of a round-robin tournament or from voting, sometimes under the name of fair bets method (González-Díaz et al., 2014) or the self-consistency rule (Laslier, 1997). In the simplest case, the transition probability matrix $P=[p_{ij}]$ is assumed to be binary representing an adjacency matrix $p_{ij} \in \{0,1\}$, e.g., representing majority voting (Brandt et al., 2016), but it has also been studied in a more general case where the matrix has continuous values $p_{ij} \in [0,1]$.

The transition probability matrix $P=[p_{ij}]$ defines a Markov chain with n states (one for each alternative) considering a random walk on the set of alternatives, and assuming the probability of visiting the next alternative does not depend on the path followed to arrive at the present alternative. According to the theory of Markov chains, a steady state row vector π, whose elements π_i represent the long-run (stationary) probability of being in the state corresponding to choosing a_i, satisfies the equilibrium equation:

$$\pi P=\pi, \text{ with } \sum_{j=1}^{n} \pi_j = 1 \text{ and } \pi_j \geq 0.$$

Under some conditions (e.g., if all p_{ij} are strictly positive) the Markov chain will be ergodic and $P^t = [p_{ij}^t]$, i.e., the t-th power of the transition matrix, converges to a matrix where each row corresponds to the stationary distribution π, i.e.,

$$\lim_{t\to\infty} p_{ij}^t = \pi_j$$

In such conditions, P is irreducible and the Markov ranking of the alternatives by decreasing order of their steady-state choice probability, π_j, is characterized by five properties, i.e., it is the unique method satisfying these properties (see González-Díaz et al. (2014) for formal definitions and proofs):

- Anonymity: a standard property requiring that the results do not dependent on the labelling of the alternatives.
- Homogeneity: results are invariant to a rescaling of P by multiplying this matrix by a positive constant.
- Symmetry: all alternatives are tied in the results if $p_{ij}=p_{ji}$, $\forall i, j$.
- Flatness preservation: if according to matrix P all alternatives are tied and the same occurs for matrix P', then all players are tied for matrix $P+P'$.
- Negative response to losses (Slutzki & Volij, 2005): if according to matrix P all alternatives are tied, then a matrix P' obtained by multiplying the losses of each alternative a_i (i.e., multiplying the i-th column of P) by a constant $\lambda_i>1$ yields a ranking inversely related to the value of these constants.

The last two properties assume P can be any non-negative matrix with null diagonal, whereas the case in which P represents transition probabilities, the elements of P have an upper bound of 1 and $\sum_{j=1}^{n} p_{ij} = 1, \forall i$. Assuming the additional condition that $p_{ij}+p_{ji}=1/n$, $\forall i, j$, Herrero & Villar (2021) presented a further application of the Markov solution for social choice, named Borda-Condorcet rule, mentioning it satisfies standard social choice methods properties: universal domain, anonymity, neutrality, weak Pareto principle, weak unanimity, independence of fully dominated alternatives, independence of generally unconcerned individuals, replication invariance, and monotonicity. The same authors also address the situation in which $p_{ij}+p_{ji}<1/n$, $\forall i, j$.

To a great extent the Markov solution approach has been studied in the contexts of voting and tournaments (Brandt et al., 2016; González-Díaz et al., 2014; Laslier, 1997). Besides voting and tournaments, this type of approach has been used for ranking websites in search engines (Brandt et al., 2016). Among ranking methods that can be related to the ideas of the Markov solution, the PageRank method and its variants have been the focus of much attention recently and applied not only to rank web pages, but also in sports, bibliometrics, and other application areas (Chung, 2014; Franceschet, 2011; Gleich, 2015; Langville & Meyer, 2012). Also, recently, Herrero & Villar (2021), suggested a way of using the Markov solution concept in a setting with multiple issues by selecting issues in a random way. To our best knowledge, no prior work has studied the concept of Markov solution as a means to exploit an outranking relation.

To address this gap, in the following section we propose and study different ways that the Markov solution can be adapted to exploit an outranking relation, through a suitable translation of the deterministic outranking relation (crisp or valued) into a stochastic outranking relation.

3. The stochastic exploitation method

Following the idea behind the Markov solution, the exploitation strategy proposed in this work consists in selecting the alternative with the highest probability of being chosen, based on a probabilistic transition matrix. In the context of tournaments, the transition probabilities are derived from the results obtained when each alternative plays against other alternatives. In the context of voting, the transition majorities are derived from the number of voters supporting each alternative against other alternatives. In the context of MCDA, Herrero & Villar (2021) suggested that multi-issue (multicriteria) evaluations could be considered by selecting a criterion randomly when comparing two options. Here, we discuss how an outranking relation can originate a transition probability matrix.

To define each transition probability, we consider that at some point alterative a_i is the current incumbent (the provisional choice of the DM) and a challenger a_j (another alternative in A, different from a_i) is randomly chosen with probability $1/(n\text{-}1)$, i.e., all other alternatives can be the challenger with the same probability. Let c_{ij} denote the probability that the DM would exchange a_i for a_j if the latter appears as a challenger. The following subsections 3.1-3.3 discuss different ways of modelling the exchange probabilities c_{ij} based on an outranking relation. The notation and characteristics of this outranking relation follow the logic of ELECTRE outranking relations (Roy, 1991), considering outranking means "at least as good as". Adaptations can be made if the outranking relation refers to a strict preference, as occurs for PROMETHEE methods

(Brans et al., 1986). Then, subsection 3.4 presents how to use these probabilities to compute the long-term (stationary) probability that each alternative is chosen.

3.1. Deterministic crisp relation

A crisp binary relation S is defined on the set of alternatives when the outranking relation either holds or does not hold for each ordered pair of alternatives. Following the usual notation in ELECTRE methods, for an ordered pair $(a_i,a_j)\in A\times A$, either a_iSa_j, meaning that a_i outranks (is at least as good as) a_j, or $\neg(a_iSa_j)$, meaning the contrary. Based on the outranking relation, three other relations can be defined as a (P,I,R) system of preference relations (B. Roy, 1996):

$a_iPa_j \Leftrightarrow a_iSa_j \wedge\neg(a_jSa_i)$ (preference relation)

$a_iIa_j \Leftrightarrow a_iSa_j \wedge a_jSa_i$ (indifference relation)

$a_iRa_j \Leftrightarrow \neg(a_iSa_j) \wedge \neg(a_jSa_i)$ (incomparability relation)

As presented above, the transition probabilities p_{ij} depend on the probability c_{ij} that the DM would exchange a_i for a_j if the latter appears as a challenger. In the context of a (P,I,R) system of preferences we envisage three working hypotheses as follows:

- *H1:* $c_{ij} = 1$ if and only if a_jPa_i (otherwise $c_{ij} = 0$). This corresponds to a strong attachment to the status quo alternative. The DM will exchange a_i for a_j only if the latter is strictly preferred to it.
- *H2:* $c_{ij} = 1$ if and only if a_jSa_i, i.e., if a_jPa_i or a_jIa_i (otherwise $c_{ij} = 0$). This corresponds to a weak attachment to the status quo alternative. The DM will exchange a_i for a_j if the latter is strictly preferred to it or if it is considered to be indifferent to it.
- *H3:* $c_{ij} = 1$ if a_jPa_i, a_jIa_i or a_jRa_i ($c_{ij} = 0$ only if a_iPa_j). This corresponds to an exploratory attitude in which the DM will exchange a_i for a_j unless a_i is strictly preferred to a_j.

3.2. Deterministic valued relation

We now examine the case when a valued binary relation S in [0,1] is defined on the set of alternatives. For an ordered pair $(a_i,a_j)\in A\times A$, let s_{ij} denote the credibility degree for the statement that a_i is at least as good as a_j. In ELECTRE methods, the exploitation of the valued outranking relation involves defining a cutting threshold λ to transform the valued outranking relation into a crisp one:

$$a_iSa_j \Leftrightarrow s_{ij}\geq\lambda$$

In the presence of a fixed value for λ, the outranking relation is crisp, as presented in the previous subsection. In the absence of a fixed value for λ, the probabilistic interpretation of the outranking relation can be grounded on a volume-based (domain-based) interpretation, equivalent to drawing λ from a uniform distribution U(0.5, 1.0), see Figure 1.

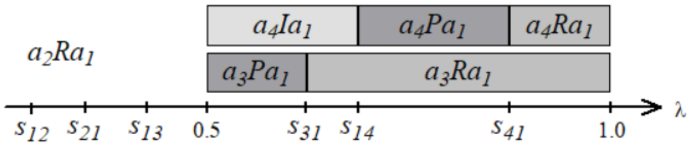


Figure 1. Preference relations as a function of the cutting level λ.

Again, three working hypotheses are envisaged for the exchange probability, analogous to the crisp case:

H1: c_{ij} is the probability of a_jPa_i

$$c_{ij} = \begin{cases} 2(s_{ji} - s_{ij}), & if\ s_{ji} > s_{ij} \wedge s_{ij} > 0.5 \\ 2(s_{ji} - 0.5), & if\ s_{ji} > 0.5 \wedge s_{ij} \leq 0.5 \\ 0, & otherwise \end{cases}$$

H2: c_{ij} is the probability of a_jSa_i

$$c_{ij} = \begin{cases} 2(s_{ji} - 0.5), & if\ s_{ji} > 0.5 \\ 0, & otherwise \end{cases}$$

H3: c_{ij} is the probability of a_jPa_i, a_jIa_i or a_jRa_i, i.e, the probability of $\neg(a_iPa_j)$

$$c_{ij} = \begin{cases} 1, & if\ s_{ji} \geq s_{ij} \vee s_{ij} < 0.5 \\ 2(1 - s_{ij}), & if\ s_{ji} < s_{ij} \wedge s_{ji} \leq 0.5 \\ 1 - 2(s_{ij} - s_{ji}), & if\ s_{ji} < s_{ij} \wedge s_{ji} > 0.5 \end{cases}$$

More generally, the same logic can be applied to any function that converts a deterministic outranking degree s_{ij} into an outranking probability c_{ij}. Such a function $F(.)$:[0,1]→[0,1] must be such that:

- $F(0)$=0, i.e., outranking is impossible if the credibility degree is null;
- $F(1)$=1, i.e., outranking is guaranteed if the credibility degree is equal to one;
- $F(.)$ is nondecreasing, i.e., if the credibility degree increases the outranking probability cannot decrease.

These properties imply that $F(.)$ can be seen as a cumulative distribution function for some probability distribution, which can be interpreted as a probability distribution for a stochastic cutting level. Indeed, the interpretation behind Figure 1 corresponds to a uniform distribution U(0.5, 1.0), but other distributions supported on a bounded interval can be used, such as the triangular, Beta, PERT, etc. (e.g., Figure 2).

The above possibilities for the exchange probability can then be rewritten as follows:

H1: c_{ij} is the probability of a_jPa_i

$$c_{ij} = F(s_{ji}) - F(s_{ij})$$

H2: c_{ij} is the probability of a_jSa_i

$$c_{ij} = F(s_{ji})$$

H3: c_{ij} is the probability of a_jPa_i, a_jIa_i or a_jRa_i, i.e, the probability of $\neg(a_iPa_j)$

$$c_{ij} = 1 - \left(F\left(s_{ij}\right) - F\left(s_{ji}\right)\right)$$

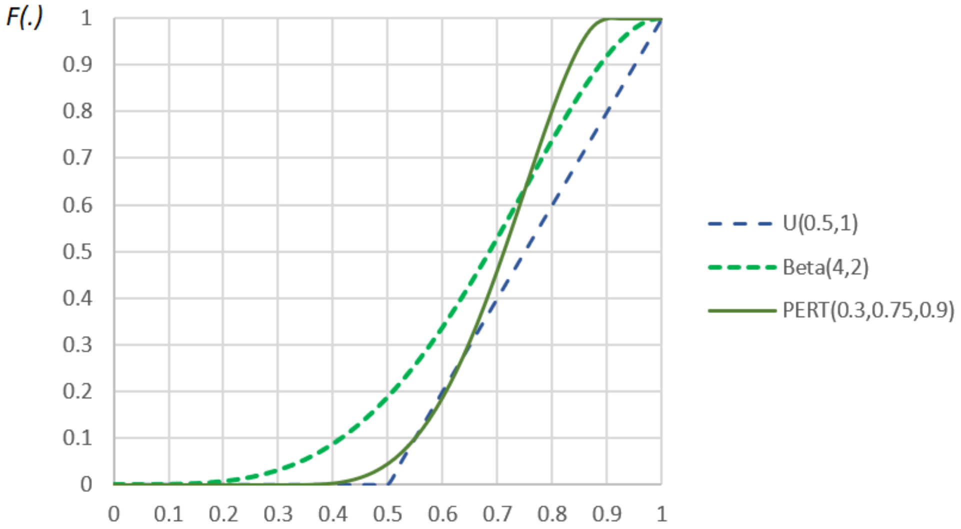


Figure 2. Credibility to probability cumulative distribution functions

3.3. Stochastic valued relation

A third possibility is that c_{ij} is given by a stochastic analysis, e.g., resulting from Monte-Carlo simulation of the outranking relation given stochastic distributions for the parameters of an MCDA outranking model, as in SMAA methods. The same three cases considered before can then be considered using the probabilities for a_jPa_i (H1), a_jSa_i (H2) or $\neg(a_iPa_j)$ (H3) resulting from the stochastic analysis:

H1: c_{ij} = Prob(a_jPa_i)

H2: c_{ij} = Prob(a_jSa_i)

H3: c_{ij} = 1-Prob(a_iPa_j)

3.4. Computation of the stationary probabilities

To define each transition probability p_{ij} we consider that when alterative a_i is the current incumbent (the provisional choice of the DM) the challenger a_j can be any other alternative, with the same probability 1/(n-1). Given c_{ij}, the probability that the DM would exchange a_i for a_j if the latter appears as a challenger, we can write

$$p_{ij} = c_{ij} /(n-1) \text{ for all } i \neq j \text{ and } p_{ii} = 1 - \sum_{j \neq i} p_{ij} \text{ for all } i.$$

(Note: it is easy to check that $p_{ii} \in [0,1]$).

All the alternatives are considered to have the same probability (1/n) of being considered in the first place. From $P=[p_{ij}]$, we then seek to obtain the steady stationary distribution $\pi = (\pi_1, \dots, \pi_n)$, with π_j representing the long-run (stationary) probability of choosing alternative a_j.

The solution, henceforth named "equiprobable start solution" (ESS), can be obtained by solving a system of equations or by simulation.

The above reasoning follows the idea of the Markov solution, assuming an infinite number of comparisons, each one potentially replacing the incumbent alternative by one of its challengers, and also assuming exchange probabilities depend solely on the outranking relation (and not on the path followed to arrive to the incumbent alternative). Of course, in practice DMs would be uncapable of performing such a large number of comparisons, but we assume they would accept to follow this logic, since the outcome is not hard to compute, as described in the following paragraphs.

A) Solution solving a system of equations

A solution can be obtained by solving the system of linear equations $\pi P=\pi$, which is guaranteed to have at least one solution (see, e.g., Chpt. 4 of Gallager (2013)). Although finding a solution is guaranteed, it may happen that the system $\pi P=\pi$ admits multiple solutions, and only one among these will correspond to the ESS. However, if P irreducible (i.e., if transition probabilities are such that a path exists to reach a_j starting from a_i, $\forall i,j$), then the system $\pi P=\pi$ admits only one solution, which is independent of the initial conditions and therefore corresponds to the ESS.

It is also possible to find the ESS with a very good approximation without needing to verify if P is irreducible. One possibility consists in using the simulation process presented below.

B) Solution using a simulation process

The ESS can be obtained by reiterating $\pi^{(t+1)}=\pi^{(t)}P$ from $t=0$ up to a large number n_{iter}, starting with $\pi^{(0)}=[\frac{1}{n}\ldots\frac{1}{n}]$, emulating a Monte-Carlo simulation process. Noting that, e.g., $\pi^{(1)}=\pi^{(0)}P$ and $\pi^{(2)}=\pi^{(1)}P$ imply $\pi^{(2)}=\pi^{(0)}P^2$, and so on, by induction this process corresponds to the approximated solution $\pi \cong \left[\frac{1}{n}\ldots\frac{1}{n}\right] P^{n_{iter}}$. In Appendix A, we show that this process is convergent.

4. Analysis of examples with three alternatives (crisp deterministic relation)

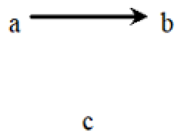
It is instructive to study the transition matrices P for a simple case with three alternatives and the ESS that correspond to each working hypothesis H1-H3 envisaged for the exchange probability. All possible combinations (apart from permutations) are depicted in Figure 3.

i) $c_{xy}=1 \Leftrightarrow yPx$ ii) $c_{xy}=1 \Leftrightarrow yPx \vee yIx$ iii) $c_{xy}=1 \Leftrightarrow yPx \vee yIx \vee yRx$

(A)

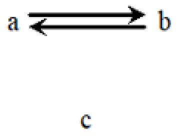
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1/3	a	1	0	0	1/3	a	0	0,5	0,5	1/3
b	0	1	0	1/3	b	0	1	0	1/3	b	0,5	0	0,5	1/3
c	0	0	1	1/3	c	0	0	1	1/3	c	0,5	0,5	0	1/3

(B)



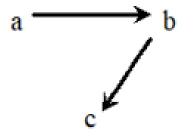
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	2/3	a	1	0	0	2/3	a	0,5	0	0,5	1/2
b	0,5	0,5	0	0	b	0,5	0,5	0	0	b	0,5	0	0,5	1/6
c	0	0	1	1/3	c	0	0	1	1/3	c	0,5	0,5	0	1/3

(C)



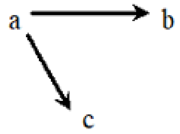
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1/3	a	0,5	0,5	0	1/3	a	0	0,5	0,5	1/3
b	0	1	0	1/3	b	0,5	0,5	0	1/3	b	0,5	0	0,5	1/3
c	0	0	1	1/3	c	0	0	1	1/3	c	0,5	0,5	0	1/3

(D)



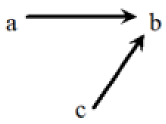
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1	a	1	0	0	1	a	0,5	0	0,5	1/2
b	0,5	0,5	0	0	b	0,5	0,5	0	0	b	0,5	0,5	0	1/4
c	0	0,5	0,5	0	c	0	0,5	0,5	0	c	0,5	0,5	0	1/4

(E)



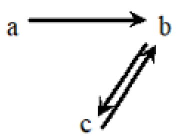
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1	a	1	0	0	1	a	1	0	0	1
b	0,5	0,5	0	0	b	0,5	0,5	0	0	b	0,5	0	0,5	0
c	0,5	0	0,5	0	c	0,5	0	0,5	0	c	0,5	0,5	0	0

(F)



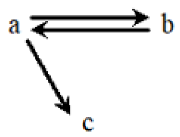
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1/2	a	1	0	0	1/2	a	0,5	0	0,5	1/2
b	0,5	0	0,5	0	b	0,5	0	0,5	0	b	0,5	0	0,5	0
c	0	0	1	1/2	c	0	0	1	1/2	c	0,5	0	0,5	1/2

(G)



P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	2/3	a	1	0	0	1	a	0,5	0	0,5	1/2
b	0,5	0,5	0	0	b	0,5	0	0,5	0	b	0,5	0	0,5	1/6
c	0	0	1	1/3	c	0	0,5	0,5	0	c	0,5	0,5	0	1/3

(H)

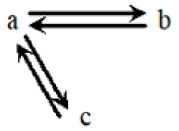


P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	2/3	a	0,5	0,5	0	1/2	a	0,5	0,5	0	1/2
b	0	1	0	1/3	b	0,5	0,5	0	1/2	b	0,5	0	0,5	1/3
c	0,5	0	0,5	0	c	0,5	0	0,5	0	c	0,5	0,5	0	1/6

Figure 3. Crisp outranking relations for three alternatives and corresponding ESS

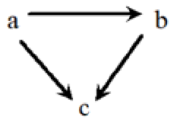
i) $c_{xy}=1 \Leftrightarrow yPx$ ii) $c_{xy}=1 \Leftrightarrow yPx \vee yIx$ iii) $c_{xy}=1 \Leftrightarrow yPx \vee yIx \vee yRx$

(I)



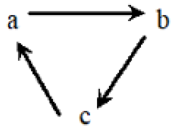
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1/3	a	0	0,5	0,5	1/3	a	0	0,5	0,5	1/3
b	0	1	0	1/3	b	0,5	0,5	0	1/3	b	0,5	0	0,5	1/3
c	0	0	1	1/3	c	0,5	0	0,5	1/3	c	0,5	0,5	0	1/3

(J)



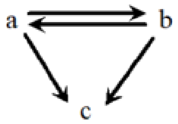
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1	a	1	0	0	1	a	1	0	0	1
b	0,5	0,5	0	0	b	0,5	0,5	0	0	b	0,5	0,5	0	0
c	0,5	0,5	0	0	c	0,5	0,5	0	0	c	0,5	0,5	0	0

(K)



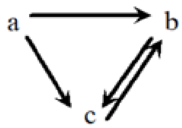
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	0,5	0	0,5	1/3	a	0,5	0	0,5	1/3	a	0,5	0	0,5	1/3
b	0,5	0,5	0	1/3	b	0,5	0,5	0	1/3	b	0,5	0,5	0	1/3
c	0	0,5	0,5	1/3	c	0	0,5	0,5	1/3	c	0	0,5	0,5	1/3

(L)



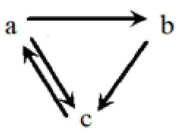
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1/2	a	0,5	0,5	0	1/2	a	0,5	0,5	0	1/2
b	0	1	0	1/2	b	0,5	0,5	0	1/2	b	0,5	0,5	0	1/2
c	0,5	0,5	0	0	c	0,5	0,5	0	0	c	0,5	0,5	0	0

(M)



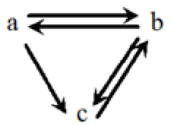
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1	a	1	0	0	1	a	1	0	0	1
b	0,5	0,5	0	0	b	0,5	0	0,5	0	b	0,5	0	0,5	0
c	0,5	0	0,5	0	c	0,5	0,5	0	0	c	0,5	0,5	0	0

(N)



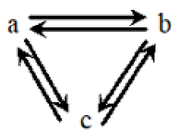
P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1	a	0,5	0	0,5	1/2	a	0,5	0	0,5	1/2
b	0,5	0,5	0	0	b	0,5	0,5	0	1/4	b	0,5	0,5	0	1/4
c	0	0,5	0,5	0	c	0,5	0,5	0	1/4	c	0,5	0,5	0	1/4

(O)



P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	2/3	a	0,5	0,5	0	1/2	a	0,5	0,5	0	1/2
b	0	1	0	1/3	b	0,5	0	0,5	1/3	b	0,5	0	0,5	1/3
c	0,5	0	0,5	0	c	0,5	0,5	0	1/6	c	0,5	0,5	0	1/6

(P)



P	a	b	c	π	P	a	b	c	π	P	a	b	c	π
a	1	0	0	1/3	a	0	0,5	0,5	1/3	a	0	0,5	0,5	1/3
b	0	1	0	1/3	b	0,5	0	0,5	1/3	b	0,5	0	0,5	1/3
c	0	0	1	1/3	c	0,5	0,5	0	1/3	c	0,5	0,5	0	1/3

Figure 3 (cont.). Crisp outranking relations for three alternatives and corresponding ESS

Situations (A), (K) and (P) are fully symmetric, and the choice probabilities given by π are naturally the same. Situations (E), (J) and (M) have a clear winner preferred to both competing alternatives, hence it is also natural to give the winner a choice probability of 1. The three possibilities to define c_{ij} also yield the same π in the following cases:

- (C) and (I), where all the relations are symmetric (I or R) and no distinctions are made;
- (F) and (L), where there is a clear loser with null chance of being selected and no reason to pick a single winner.

The differences appear in six cases that are harder to exploit: (B), (D), (G), (H), (N), and (O). Variant iii) corresponding to H3 is the one that yields a final distribution π more evenly spread. This can be seen as a disadvantage, since it provides a less clear-cut result concerning the aim of identifying the most preferred alternative. In cases (B) and (H), but also (G), it is difficult to understand why should anyone contemplate choosing the alternative with choice probability 1/6, whereas variants i) (H1) and ii) (H2) yield a more natural choice probability of 0. We therefore conclude that H1 and H2 are preferable to H3. For this reason, in the continuation we drop variant iii) (H3) and focus on H1 and H2.

The kernel method could also be used to analyze many of these relations. In some situations, the results would be perfectly aligned:

- In situation (A) the three alternatives would be in the kernel, which matches well having the same choice probabilities;
- In situations (E), (J), and (M) the kernel has a single winner, which matches well the choice probability of 1;
- In situation (F) the two outranking alternatives *a* and *c* would be in the kernel, which matches well having 0.5 choice probabilities for each one.

In other situations, one would find differences, showing the kernel method and the ESS follow different philosophies:

- In situations (B) and (G) both *a* and *c* would be in the kernel. These are the alternatives with positive choice probability (in H1 and H2). Some DMs might appreciate the discrimination provided by the ESS, with higher probability for *a* that results from its comparison with *b*, whereas other DMs might prefer the kernel method's more neutral stance of not making a distinction between *a* and *c*;
- In situation (D) the kernel consists of *a* and *c*. This reflects a cautious stance of keeping not only *a* (the only alternative not outranked), but also *c* as it is not outranked by *a*. This reflects the intransitivity of the outranking relation. The ESS, on the other hand, follows a different logic because the intuitive idea of successively exchanging the provisional choice entails some transitivity: if given *c*, the DM would exchange it for *b*, and if given *b* the DM would exchange it for *a*, this leads to the final choice of *a*;
- In situations (H) and (N), only *a* would be in the kernel, and *a* has the highest probability of the ESS. For H1 the probability of choosing *a* is greater than that for H2, given the latter's treatment of the indifference relation.

In situation (K) no kernel exists. In all the remaining situations, more than one kernel exists.

As a note, some of these cases with three alternatives are instructive to remark that the ESS (and the same applies to the kernel solution) is mostly appropriate to the context of selecting a single alternative, and not to contexts of selecting multiple alternatives or ranking the alternatives (even though the ESS probabilities could be used to define a ranking). An example is case (J), for which the ESS yields probability 1 for alternative *a* and probability 0 for *b* and *c*. The ESS makes no difference between *b* and *c* for the purpose of being the winner, as both are obviously not a winner, even though in terms of a ranking one would easily accept that after *a*, *b* would be second and *c* would be third. Another example is (B), for which the ESS under H1 or H2 indicates *b* has zero probability of being the best one, whereas *c* has some probability of being the best one. Suppose *a* has good quality and it is cheap, *b* has also good quality but it costs a bit more (hence it is outranked by *a*), and *c* has excellent quality but costs a lot more (hence it was considered to be incomparable to *a* and *b*). If the DM needed to select one alternative, *b* would obviously not be selected. However, if the DM needed to select two alternatives, then some DMs might prefer to have *a* and *b* rather than *a* and *c*.

5. Additional examples

This section revisits two real-world case studies described in the literature. The results were obtained using an exact approach solving $\pi P=\pi$.

5.1. Probabilities derived from crisp or valued relations

First, let us consider a real-world application at the French postal services described by Roy & Bouyssou (1993, Chpt. 8), which aimed at the selection of a machine to sort packages. Table 1 presents a valued outranking relation considering the concordance values obtained by the ELECTRE IS method, with zeroes corresponding to veto situations. In the cited study, this relation was exploited considering a cutting level λ=0.7, obtaining a choice set (using the kernel method) containing alternatives a_1, a_7, and a_9 as candidates to be the preferred alternative.

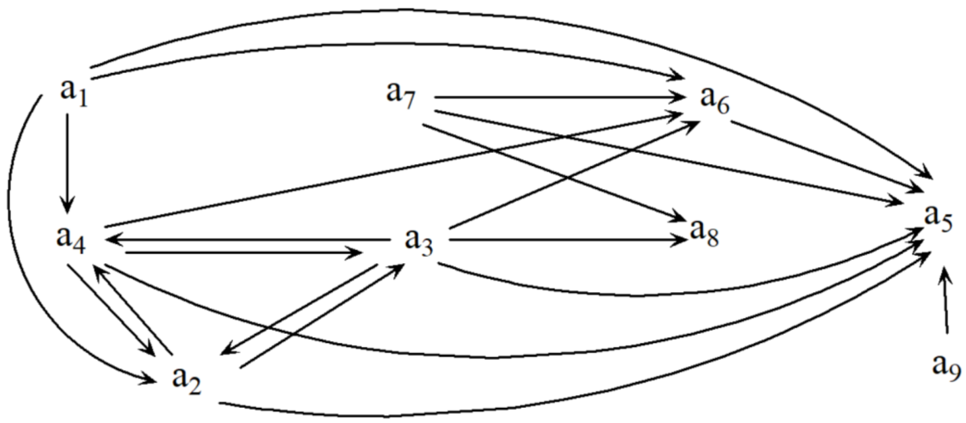


Figure 4. Crisp outranking relation $S=[s_{ij}]$ adapted from (Roy & Bouyssou, 1993).

First, let us consider only the crisp outranking relation corresponding to the cutting level λ=0.7 (Figure 4 and boldface in Table 1), i.e., $a_iSa_j \Leftrightarrow s_{ij}\geq\lambda$. The transition probabilities and the ESS corresponding to hypotheses H1 and H2 are presented in Table 2 and Table 3, respectively.

According to H1, the DM is strongly attached to the status quo alternative, and will exchange a_i for a_j only if the latter is strictly preferred to it. Then, once the DM reaches a_1, a_3, a_7, or a_9, no further exchanges occur and this provisional alternative remains the final choice. The steady state probabilities reflect the chances of ending the chain of exchanges in each of these alternatives, when the starting alternative is picked randomly, favoring alternative a_1 as the most likely to be chosen.

Table 1. Valued outranking relation S=[s_{ij}] adapted from (Roy & Bouyssou, 1993). Values in boldface correspond to the crisp outranking relation considering a cutting level λ=0.7.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
a_1	**1.000**	**0.718**	0.667	**0.795**	**0.925**	**0.744**	0.692	0	0
a_2	0.692	**1.000**	**0.872**	**0.718**	**0.923**	0.641	0	0	0.667
a_3	0.615	**0.923**	**1.000**	**0.846**	**0.923**	**0.769**	0	**0.795**	0.59
a_4	0.692	**0.846**	**0.795**	**1.000**	**1.000**	**0.718**	0	0	0
a_5	0	0	0	0	**1.000**	0	0	0	0
a_6	0	0	0	0	**0.949**	**1.000**	0.615	0	0
a_7	0	0.590	0.538	0.615	**0.872**	**0.923**	**1.000**	**0.795**	0
a_8	0.538	0	0	0	0.667	0.590	0.590	**1.000**	0
a_9	0.436	0	0	0	**0.744**	0	0	0.615	**1.000**

Table 2. Transition matrix P and corresponding ESS for the crisp relation under hypothesis H1.

P	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	*ESS*
a_1	1.000	0	0	0	0	0	0	0	0	0.444
a_2	0.125	0.875	0	0	0	0	0	0	0	0
a_3	0	0	1.000	0	0	0	0	0	0	0.214
a_4	0.125	0	0	0.875	0	0	0	0	0	0
a_5	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0	0.125	0
a_6	0.125	0	0.125	0.125	0	0.500	0.125	0	0	0
a_7	0	0	0	0	0	0	1.000	0	0	0.214
a_8	0	0	0.125	0	0	0	0.125	0.750	0	0
a_9	0	0	0	0	0	0	0	0	1	0.127

Table 3. Transition matrix P and corresponding ESS for the crisp relation under hypothesis H2.

P	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	ESS
a_1	1.000	0	0	0	0	0	0	0	0	0.659
a_2	0.125	0.625	0.125	0.125	0	0	0	0	0	0
a_3	0	0.125	0.75	0.125	0	0	0	0	0	0
a_4	0.125	0.125	0.125	0.625	0	0	0	0	0	0
a_5	0.125	0.125	0.125	0.125	0.125	0.125	0.125	0	0.125	0
a_6	0.125	0	0.125	0.125	0	0.599	0.125	0	0	0
a_7	0	0	0	0	0	0	1.000	0	0	0.214
a_8	0	0	0.125	0	0	0	0.125	0.750	0	0
a_9	0	0	0	0	0	0	0	0	1	0.127

According to H2, the DM is not so attached to the status quo alternative and will exchange a provisional alternative a_i for a_j if the latter is at least as good as the provisional alternative. Thus, if the provisional alternative is a_3, the DM might exchange it for a_2 or a_4, and the exchange chain eventually ends up at a_1, a_7 or a_9. Coincidentally, these are the three alternatives in the kernel solution, but the ESS provides an argument to select a single one. H2 benefits a_1, which increases the probability of being chosen to 0.659.

Let us now consider the valued outranking relation in Table 1, i.e., without defining a value for the cutting level. Following the method proposed in Section 3.2, this requires specifying a credibility-to-probability function F. In this example we consider hypotheses H1 and H2 and two different functions: U(0.5, 1.0), following the rationale in Figure 1, and Pert(0.3, 0.75, 0.9). Both functions are depicted in Figure 2.

Results are presented in Tables 4-7. Tables 4 and 5 consider H1 for two distinct distribution functions, obtaining similar results. The Pert distribution allows a small probability p_{19}=0.001 (Table 5) that a_9 is preferred to a_1, even though s_{91}=0.436<0.5 (Table 1), whereas according to U(0.5, 1.0) this preference would never occur. Comparing Tables 4 or 5 with Table 2 (crisp relation), differences are much larger. Alternative a_1 is still the most likely to be chosen, but with much higher probability when exploiting the valued relation directly instead of using λ=0.7. The valued relation benefits a_1 mainly because its outranking credibility over a_7 is s_{17}=0.692, which is below but very near the cutting level. The stationary probability of selecting a_7 is thus much lower in Tables 4 and 5. Another alternative that sees its choice probability decrease much is a_9. The reason is similar: s_{29}=0.667 and s_{39}=0.590, both below λ, enabling transition from a_9 to a_2 or a_3 when exploiting the outranking relation, which was not possible before. Comparing Tables 4 or 5 with Table 2 also indicates the exploitation of the crisp relation yields more alternatives with null probability of being chosen in the stationary distribution. Since the exploitation of the valued relation takes into account differences between s_{ij} and s_{ji} when both are below (or above) the cutting level of the crisp relation, small transition probabilities appear. Nevertheless, the alternatives with null probability of being chosen in Table 2 still have a very small choice probability in Tables 4 and 5.

Table 4. Transition matrix P and corresponding ESS for the valued relation considering function $U(0.5, 1.0)$ *and hypothesis H1.*

P	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	ESS
a_1	0.991	0	0	0	0	0	0	0.01	0	0.675
a_2	0.007	0.926	0.013	0.032	0	0	0.023	0	0	0.011
a_3	0.013	0	0.978	0	0	0	0.01	0	0	0.164
a_4	0.026	0	0.013	0.933	0	0	0.029	0	0	0.005
a_5	0.106	0.106	0.106	0.125	0.249	0.112	0.093	0.042	0.061	0.000
a_6	0.061	0.035	0.067	0.055	0	0.683	0.077	0.023	0	0.000
a_7	0.048	0	0	0	0	0	0.952	0	0	0.085
a_8	0	0	0.074	0	0	0	0.051	0.846	0.029	0.042
a_9	0	0.042	0.023	0	0	0	0	0	0.936	0.019

Table 5. Transition matrix P and corresponding ESS for the valued relation considering function $Pert(0.3, 0.75, 0.9)$ *and hypothesis H1.*

P	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	ESS
a_1	0.988	0	0	0	0	0	0	0.011	0.001	0.672
a_2	0.011	0.914	0.002	0.051	0	0	0.021	0	0.000	0.015
a_3	0.017	0	0.972	0	0	0	0.011	0	0.000	0.152
a_4	0.044	0	0.018	0.91	0	0	0.028	0	0.000	0.009
a_5	0.125	0.125	0.125	0.125	0.131	0.125	0.123	0.045	0.076	0.000
a_6	0.076	0.036	0.087	0.065	0	0.617	0.097	0.021	0.000	0.000
a_7	0.054	0	0	0	0	0	0.946	0	0.000	0.089
a_8	0	0	0.098	0	0	0	0.078	0.796	0.028	0.035
a_9	0	0.045	0.021	0	0	0	0	0	0.934	0.028

Tables 6 and 7 are the equivalent of Tables 4 and 5 considering H2 for the same distribution functions. As occurred for H1, the choice of the distribution did not impact much the results. In turn, comparing Tables 6 or 7 with Table 3 (crisp relation), differences are much larger. Curiously, under H2, the exploitation of the valued relation makes it less clear that a_1 should be chosen, although it is still the most likely one. Under H2, a_2, a_3, and a_4 have a relatively high credibility of outranking a_1, respectively 0.692, 0.615, and 0.692, which was not considered in the crisp relation because it is lower than the cutting level. Thus, now it becomes possible to transition from a_1 to a_2, a_3, or a_4. Note that this does not occur under H1, because although s_{21}, s_{31}, and s_{41} are relatively high, s_{12}, s_{13}, and s_{14} are even higher and therefore the preference relation occurs only in favor of a_1. Similarly to what is observed under H1, the direct exploitation of the valued outranking relation yields fewer alternatives with null choice probability.

Based on either H1 or H2, which reflect two different perspectives concerning the willingness to exchange an alternative for another one, the Markov solution provides a rationale to select one of the alternatives. In this case, rather than having a kernel of several undistinguished alternatives to choose from, the Markov solution typically will point to a single one as having the maximum probability of being chosen.

Table 6. Transition matrix P and corresponding ESS for the valued relation considering function U(0.5, 1.0) *and hypothesis H2.*

P	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	ESS
a_1	0.866	0.048	0.029	0.048	0	0	0	0.01	0	0.274
a_2	0.055	0.731	0.106	0.087	0	0	0.023	0	0	0.155
a_3	0.042	0.093	0.782	0.074	0	0	0.01	0	0	0.196
a_4	0.074	0.055	0.087	0.757	0	0	0.029	0	0	0.171
a_5	0.106	0.106	0.106	0.125	0.249	0.112	0.093	0.042	0.061	0.000
a_6	0.061	0.035	0.067	0.055	0	0.654	0.106	0.023	0	0.012
a_7	0.048	0	0	0	0	0.029	0.901	0.023	0	0.142
a_8	0	0	0.074	0	0	0	0.074	0.824	0.029	0.034
a_9	0	0.042	0.023	0	0	0	0	0	0.936	0.015

Table 7. Transition matrix P and corresponding ESS for the valued relation considering function $Pert$(0.3, 0.75, 0.9) *and hypothesis H2.*

P	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	ESS
a_1	0.852	0.054	0.028	0.054	0	0	0	0.011	0.001	0.294
a_2	0.065	0.672	0.125	0.117	0	0	0.021	0	0.000	0.157
a_3	0.045	0.123	0.724	0.098	0	0	0.011	0	0.000	0.187
a_4	0.098	0.065	0.117	0.692	0	0	0.028	0	0.000	0.173
a_5	0.125	0.125	0.125	0.125	0.131	0.125	0.123	0.045	0.076	0.000
a_6	0.076	0.036	0.087	0.065	0	0.589	0.125	0.021	0.000	0.009
a_7	0.054	0	0	0	0	0.028	0.897	0.021	0.000	0.135
a_8	0	0	0.098	0	0	0	0.098	0.776	0.028	0.027
a_9	0	0.045	0.021	0	0	0	0	0	0.934	0.018

5.2. Probabilities given by a stochastic relation

The Markov solution can also be used when the outranking or preference probabilities have already been computed by some other means. For instance, Govindan et al. (2019) analyzed the problem of choosing a reverse logistics provider for an Indian automotive company. To this aim, the authors performed a Stochastic Multicriteria Acceptability Analysis (SMAA) considering an ELECTRE underlying model. Accounting for uncertainty in the preference-related parameters, the results of SMAA included a matrix of outranking probabilities (the outranking acceptability indices) and a matrix of preference probabilities (the preference acceptability indices), both depicted in Table 8. According to SMAA, the probabilities provided in Table 8 have been obtained from a Monte-Carlo simulation of the crisp outranking relation that results for many thousands of randomly generated parameter values for an ELECTRE preference model.

Table 8. Stochastic preference and outranking relations from Govindan et al. (2019).

	Preference acceptability indices					Outranking acceptability indices				
	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
a_1	0.000	0.303	0.787	0.653	0.000	1.000	1.000	0.787	0.974	0.000
a_2	0.000	0.000	0.726	0.317	0.000	0.697	1.000	0.726	0.494	0.000
a_3	0.000	0.000	0.000	0.000	0.000	0.000	0.000	1.000	0.000	0.000
a_4	0.000	0.000	0.587	0.000	0.000	0.321	0.177	0.587	1.000	0.124
a_5	0.592	0.546	1.000	0.876	0.000	0.592	0.546	1.000	1.000	1.000

Under hypothesis H1, we consider the preference acceptability indices, leading to the transition probabilities and the ESS in Table 9. In this case, the transitions invariably lead to choose a_5. Indeed, all alternatives have a positive probability of transitioning to a_5, and once they arrive there is no other alternative preferred to it.

Under hypothesis H2, we consider the outranking acceptability indices, leading to the transition probabilities and the ESS in Table 10. Alternative a_5 is the most likely to be chosen, but the small chance that it is outranked by a_4 allows other alternatives (except a_3) to appear with a positive stationary probability. Yet, this probability is very small, and does not put into question the superiority of a_5. These conclusions match those obtained by Govindan et al. (2019) through a simulation of which alternatives might appear in the kernel.

Table 9. Transition matrix P and corresponding ESS for the stochastic preference relation considering (hypothesis H1).

	a_1	a_2	a_3	a_4	a_5	*ESS*
a_1	0.852	0	0	0	0.148	0
a_2	0.076	0.788	0	0	0.137	0
a_3	0.197	0.182	0.225	0.147	0.25	0
a_4	0.163	0.079	0	0.539	0.219	0
a_5	0	0	0	0	1.000	1

Table 10. Transition matrix P and corresponding ESS for the stochastic preference relation considering (hypothesis H2).

	a_1	a_2	a_3	a_4	a_5	*ESS*
a_1	0.598	0.174	0	0.08	0.148	0.055
a_2	0.250	0.569	0	0.044	0.137	0.037
a_3	0.197	0.182	0.225	0.147	0.250	0
a_4	0.244	0.124	0	0.383	0.250	0.053
a_5	0	0	0	0.031	0.969	0.854

6. Properties of the stochastic exploitation method

This section addresses two important properties of the stochastic exploitation process proposed in this work. The Markov solution has been studied in other contexts (see, e.g., González-Díaz et al. (2014)), but some properties and proofs that apply to tournaments do not apply or need to be redefined for our specific context. Considering all the variants introduced in Section 3, we find the following characteristics in common:

- s_{ij} and s_{ji} can take any values in [0,1], $\forall i,j$. No constraint on their sum is considered, such as the usual assumptions $s_{ij}+s_{ji}=1$ or $s_{ij}+s_{ji}\leq 1$.

- p_{ij}, $i\neq j$, have an upper bound $1/(n-1)$.

- p_{ij}, the probability of the transition from a_i to a_j, cannot increase with s_{ij} and cannot decrease with s_{ji}, i.e., for any two outranking relations S and S', $s'_{ij} > s_{ij} \Rightarrow p'_{ij} \leq p_{ij} \wedge p'_{ji} \geq p_{ji}, \forall i \neq j$ (note the right-side inequalities are not strict). No relation can be assumed between p_{ij} and p_{ji}.

The methods proposed here lead to a transition matrix P with the property that $\sum_{j=1}^{n} p_{ij} = 1, \forall i$ and $p_{ij} \in [0, 1/(n-1)]$ $\forall i \neq j$, i.e., not restricted to $p_{ij}+p_{ji}\leq 1/(n-1)$, $\forall i,j$ as in (Herrero & Villar, 2021), but less general than the case characterized by González-Díaz et al. (2014).

The ESS obviously satisfies typical the properties of Anonymity and Symmetry included in González-Díaz et al. (2014)'s characterization, whereas the properties of Homogeneity and

Flatness preservation, and Slutzki & Volij (2005)'s Negative response to losses are not applicable in our context where adding outranking relations or multiplying them by a constant is meaningless. More generally than Negative response to losses, we next define and prove a monotonicity property, and we also analyze a property based on dominance. These properties have already been proved for more specific contexts, namely for a transition matrix P with the property $p_{ij}+p_{ji}\leq 1/(n-1)$, $\forall i,j$ (Herrero & Villar, 2021), but they remain to be demonstrated for our more general context and also expressed in terms of the performances of the alternatives on multiple criteria and the way it impacts the outranking relation.

6.1. Monotonicity

Monotonicity is a common desideratum in MCDA. This property guarantees that an alternative cannot be worse-off in its evaluation if its performance becomes better on some criterion or criteria (while maintaining its performance on the remaining criteria). For instance, if the MCDA is being used to choose a vendor and one of the criteria considered is the cost, then the method should ensure that an alternative cannot be worse-off if its cost decreases (everything else remaining equal). Proposition 1 below shows the ESS respects monotonicity concerning the alternatives performances, based on the following lemma.

Lemma 1.

Let π be the ESS for an irreducible transition matrix P. Let P' be a new matrix resulting from updating P as follows, for some δ, some row w and some column b ($w \neq b$)

$p'_{wb} = p_{wb} + \delta$, with $\delta \in \left]max\{-p_{wb}, p_{ww}-1\}, min\{1-p_{wb}, p_{ww}\}\right[$

$p'_{ww} = p_{ww} - \delta$ (so that the row sum is still equal to 1)

$p'_{ij} = p_{ij}$ for all other $(i,j) \neq (w,b)$

Then, it must hold that:

- If $\delta>0$, then $\pi'_b \geq \pi_b$.
- If $\delta<0$, then $\pi'_b \leq \pi_b$.

<u>Proof</u>

Suppose that the transition probability p_{ww} is decreased by δ while p_{wb} is increased by the same amount. Assuming the resulting matrix is irreducible (managing δ bounds), then expressions for the difference in the stationary probabilities are given by [Thm 4.2. Hunter, 2005]:

$$\pi_j - \pi'_j = \begin{cases} \delta\pi_w\pi'_w m_{bw}, & j = w \\ -\delta\pi_b\pi'_w m_{wb}, & j = b \\ \delta\pi_j\pi'_w(m_{bj} - m_{wj}), & j \neq w, b \end{cases},$$

where m_{ij} is the mean time for the DM to exchange, for the first time, alternative a_i with alternative a_j. As $\pi_i . \pi'_i . m_{ij} \geq 0 \ \forall i,j$, if $\delta > 0$ then

$$\pi_w - \pi'_w \geq 0 \Rightarrow \pi'_w \leq \pi_w$$

$$\pi_b - \pi'_b \le 0 \Rightarrow \pi'_b \ge \pi_b$$
$$\pi_j - \pi'_j \ge 0 \; if \; m_{bj} - m_{wj} \ge 0 \Rightarrow \pi'_j \le \pi_j \; if \; m_{bj} \ge m_{wj}.$$

The last inequality means that regardless of the perturbations at a_w and a_b, the stationary distributions at $j \neq w, b$ depends on the "distance" a_j is from a_w and a_b.

Moreover, under the conditions of Theorem 4.2 [Corollary 4.2.2. Hunter, 2005]:

$$-\delta\pi'_w m_{wb} = \frac{\pi_b - \pi'_b}{\pi_b} \le \frac{\pi_j - \pi'_j}{\pi_j} \le \frac{\pi_w - \pi'_w}{\pi_w} = \delta\pi'_w m_{bw}, 1 \le j \le n.$$

Thus,

$$1 - \delta\pi'_w m_{bw} = \frac{\pi'_w}{\pi_w} \le \frac{\pi'_j}{\pi_j} \le \frac{\pi'_b}{\pi_b} = 1 + \delta\pi'_w m_{wb}$$

and $\frac{\pi'_w}{\pi_w} < 1 < \frac{\pi'_b}{\pi_b}$ so that $\pi'_w < \pi_w$ and $\pi'_b > \pi_b$.

Therefore, if $p'_{ww} < p_{ww}$ and $p'_{wb} > p_{wb}$ then $\pi'_w < \pi_w$ and $\pi'_b > \pi_b$. Furthermore, while the stationary probabilities at all other alternatives increase or decrease depending on the "distance" from a_w and a_b, the relative change in magnitude at any alternative never exceeds the relative changes of alternatives a_w and a_b, i.e., the minimal and maximal relative changes occur at alternatives a_w and a_b, respectively.

Proposition 1 (Monotonicity). Let *S*=[s_{ij}] be any outranking relation (crisp or valued).

i) Let *S'*=[s'_{ij}] be another relation obtained by improving the performance of some alternative a_b on one or more criteria, without any further changes. Then the steady state probability π'_b of choosing a_b either increases or remains unchanged.

ii) On the contrary, if *S'*=[s'_{ij}] is obtained by worsening the performance of some alternative a_w on one or more criteria, without any further changes, the steady state probability π'_w of choosing a_w either decreases or remains unchanged.

Proof

We will prove the first part only, as the proof for the second part is similar. The idea of this proof is that originally the outranking relation is *S*, and this relation is modified to S' in a way favorable to alternative a_b. We assume a_b has improved its performance on some criteria, and as a consequence the degree to which a_b outranks other alternatives cannot be lower than it was before, and the degree to which other alternatives outrank a_b cannot be higher than it was before.

We take for granted that, as occurs in ELECTRE methods, improving the performance of a_b implies $s'_{bj} \ge s_{bj}$ and $s'_{jb} \le s_{jb}, \forall j$. Then, noting that all the variants proposed in Section 3 would lead to $c'_{bj} \le c_{bj}$ and $c'_{jb} \ge c_{jb}, \forall j$ (with *C*=[c_{ij}] and *C*'[c'_{ij}] denoting the exchange probabilities for *S* and *S'*), we conclude that $p'_{bj} \le p_{bj}$ and $p'_{jb} \ge p_{jb}, \forall j$ (with *P*=[p_{ij}] and *P*'[p'_{ij}] denoting the transition probabilities for *S* and *S'*).

If $\nexists j \neq b: p'_{bj} < p_{bj} \vee p'_{jb} > p_{jb}$, then $\pi'_b = \pi_b$. Otherwise, by successively applying Lemma 1 to all indices $\{j \neq b$: $p'_{bj} < p_{bj} \vee p'_{jb} > p_{jb}\}$, we will successively increase the steady state probability for a_b, leading to $\pi'_b > \pi_b$.

6.2. Dominance

Dominance is also an important property to guarantee in a choice problem. In multicriteria evaluations, an alternative is said to dominate another alternative if its performance is equal to or better than the latter on all the criteria, and strictly better in at least one of them. The traditional dominance property requires that a dominated alternative is not chosen as being the best one.

In outranking methods such as ELECTRE, it may occur that an alternative a_b is considered to be indifferent to another one a_d, even if the former dominates the latter. This can occur because the current versions of such methods accept imprecision in the assessment of performance, and allow the DM to indicate indifference thresholds for performance differences in the multiple criteria (Roy, 1996). Therefore, it is possible that one alternative strictly dominates another one, but if their performance differences are below the indifference thresholds, they will outrank each other. This means that a dominance relation might not be reflected in the outranking relation, and therefore might remain invisible to the ESS method (and to any other method exploiting the same outranking relation).

For this reason, we introduce a novel concept of dominance, requiring that the dominance relation impacts the transition probabilities, i.e., not only an alternative dominates another one, but their performance differences are significant enough so that their transition probabilities are not exactly the same. This new dominance relation, which refers to the transition probabilities and not strictly to the alternatives' performance measurements, is defined as follows:

Definition: p-dominance relation

An alternative a_b p-dominates another alternative a_d if and only if all of the following holds:

i. $p_{bi} \leq p_{di}, \forall i \neq b, d$, i.e., the probability of leaving to any third alternative a_i is not lower for a_d than it is for a_b;
ii. $p_{ib} \geq p_{id}, \forall i \neq b, d$, i.e., if the current alternative is a_i, the probability of leaving to a_b is not lower than the probability of leaving to a_d;
iii. $p_{bd} < 0.5 \times\ 1/(n-1)$, i.e., if the current alternative is a_b, and if a_d is randomly selected as a potential exchange (with probability $1/(n-1)$), then the probability of accepting the exchange (leaving) is less than the probability of rejecting the exchange (staying);
iv. $p_{db} > 0.5 \times\ 1/(n-1)$, i.e., if the current alternative is a_d, and if a_b is randomly selected as a potential exchange (with probability $1/(n-1)$), then the probability of accepting the exchange (leaving) is greater than the probability of rejecting the exchange (staying);

Based on the following Lemma, we will show the ESS respects the p-dominance relation.

Lemma 2. Let a_b and a_d be two alternatives such that a_b p-dominates a_d and let the starting probability vector be $\pi^{(0)} = \left[\pi_1^{(0)} \cdots \pi_b^{(0)} \cdots \pi_d^{(0)} \cdots \pi_n^{(0)}\right]$ such that $\pi_b^{(0)} \geq \pi_d^{(0)}$ and $\pi_b^{(0)} > 0$.Then, the steady state probability vector is such that $\pi_b > \pi_d$.

Proof

Let

$\pi^{(1)} = \pi^{(0)}P = \left[\pi_1^{(1)} \cdots \pi_b^{(1)} \cdots \pi_d^{(1)} \cdots \pi_n^{(1)}\right]$.

Then,

$$\pi_b^{(1)} = \pi_1^{(0)}p_{1b} + \pi_2^{(0)}p_{2b} + \cdots + \pi_b^{(0)}p_{bb} + \cdots + \pi_d^{(0)}p_{db} + \cdots + \pi_n^{(0)}p_{nb}$$

and

$$\pi_d^{(1)} = \pi_1^{(0)}p_{1d} + \pi_2^{(0)}p_{2d} + \cdots + \pi_b^{(0)}p_{bd} + \cdots + \pi_d^{(0)}p_{dd} + \cdots + \pi_n^{(0)}p_{nd}.$$

Then, we can write

$$\pi_b^{(1)} - \pi_d^{(1)} = \left[\sum_{i \neq b,d} \pi_i^{(0)}(p_{ib} - p_{id})\right] + \pi_b^{(0)}p_{bb} + \pi_d^{(0)}p_{db} - \pi_b^{(0)}p_{bd} - \pi_d^{(0)}p_{dd}$$

As $p_{ib} \geq p_{id}, i \neq b, d$, the sum in square brackets must be non-negative, i.e.,

$$\pi_b^{(1)} - \pi_d^{(1)} \geq \Delta = \pi_b^{(0)}p_{bb} + \pi_d^{(0)}p_{db} - \pi_b^{(0)}p_{bd} - \pi_d^{(0)}p_{dd}$$

Let us now examine the sign of Δ, the right part of the inequality.

Note that $p_{bi} \leq p_{di}, \forall i \neq b, d \ \wedge \ p_{bd} < \frac{0.5}{(n-1)} < p_{db}$ imply that

$$\sum_{i \neq b} p_{bi} < \sum_{i \neq d} p_{di} \Leftrightarrow -\sum_{i \neq b} p_{bi} > -\sum_{i \neq d} p_{di} \Leftrightarrow 1 - \sum_{i \neq b} p_{bi} > 1 - \sum_{i \neq d} p_{di} \Leftrightarrow p_{bb} > \ p_{dd}.$$

Moreover, $p_{bd} < \frac{0.5}{(n-1)}$ implies

$$p_{bb} = 1 - \sum_{i \neq bd} p_{bi} - p_{bd} > 1 - \sum_{i \neq b,d} \frac{1}{(n-1)} - \frac{0.5}{(n-1)} = \frac{0.5}{(n-1)} \Leftrightarrow p_{bb} > p_{bd}.$$

Since $p_{bb} > \ p_{dd}$, $p_{bb} > p_{bd}$, and $p_{db} > p_{bd}$, the only possible rankings for these four probabilities are:

a) $p_{bb} \geq p_{db} \geq p_{bd} \geq \ p_{dd}$
b) $p_{bb} \geq p_{db} \geq \ p_{dd} \geq p_{bd}$
c) $p_{bb} \geq p_{dd} \geq p_{db} \geq \ p_{bd}$
d) $p_{db} \geq p_{bb} \geq p_{bd} \geq \ p_{dd}$
e) $p_{db} \geq p_{bb} \geq \ p_{dd} \geq p_{bd}$

Rewriting Δ as $\Delta = \pi_b^{(0)}(p_{bb} - p_{bd}) + \pi_d^{(0)}(p_{db} - p_{dd})$, and recalling $p_{bb} > p_{bd}$ it follows that in cases a), b), d) and e) $p_{bb} - p_{bd} > 0$ and $p_{db} - p_{dd} \geq 0$, thus $\pi_b^{(1)} > \pi_d^{(1)}$.

In the remaining case, $\Delta = \pi_b^{(0)} p_{bb} + \pi_d^{(0)} p_{db} - \pi_b^{(0)} p_{bd} - \pi_d^{(0)} p_{dd}$ together with $p_{bb} \geq p_{dd}$ and $p_{db} > p_{bd}$, imply

$$\Delta > \pi_b^{(0)} p_{dd} + \pi_d^{(0)} p_{db} - \pi_b^{(0)} p_{db} - \pi_d^{(0)} p_{dd} \Leftrightarrow \Delta > \left(\pi_b^{(0)} - \pi_d^{(0)}\right)(p_{dd} - p_{db})$$

Since the right-hand side of this inequality is non-negative, the conclusion $\pi_b^{(1)} > \pi_d^{(1)}$ holds.

By induction in t, assume that $\pi_b^{(t)} \geq \pi_d^{(t)}$.

Then, replacing in the reasoning above 0 by t and 1 by t+1 it is straightforward to conclude $\pi_b^{(t+1)} > \pi_d^{(t+1)}$.

Therefore, in the steady state one must have $\pi_b > \pi_d$.

Proposition 2 (p-dominance). Let a_b and a_d be two alternatives such that a_b p-dominates a_d. Then the ESS is such that $\pi_b > \pi_d$.

Proof

Since the starting probability vector for the ESS is $\pi^{(0)} = \left[\pi_1^{(0)} \cdots \pi_b^{(0)} \cdots \pi_d^{(0)} \cdots \pi_n^{(0)}\right] = \left[\frac{1}{n} \quad \dots \quad \frac{1}{n}\right]$, which fulfills $\pi_b^{(0)} \geq \pi_d^{(0)}$ and $\pi_b^{(0)} > 0$, the result follows from Lemma 2.

7. Conclusions

This work adds the Markov solution concept proposed in the context of tournaments and social choice to the toolbox of MCDA outranking methods, proposing methods to exploit an outranking relation, aiming to inform DMs in choice problems. This exploitation strategy can be applied to crisp outranking relations, valued outranking relations, or stochastic outranking relations. The proposed ESS solution can be computed without much effort by solving a system of equations or through Monte-Carlo simulation. Compared to the well-known kernel solution for crisp outranking relations, the ESS has an increased ability to pick a single winner and does not require the relation to be acyclic. The kernel solution is able to find solutions based on the transformation of the relation with cycles in indifference classes, but that might sometimes lead to results violating monotonicity, a property that ESS has been shown to comply with. On the other hand, the kernel solution follows a different logic, and can be considered more cautious as it identifies all potential candidates without directing the DM to choose one of them. Therefore, the ESS should be seen as an additional tool for the outranking methods toolbox, rather than a replacement for the existing tools.

Comparing the results of the three working hypotheses considered in this work to obtain a transition matrix, the analysis of simple cases with three alternatives allowed us to exclude the hypothesis that treated the indifference and incomparability relations in a similar way (H3). The analysis of these cases and two other cases based on real-world data (Section 5) does not allow any definitive conclusions on whether it is better to define transitions based on the preference relation *P* (H1) or the outranking relation *S* (H2). With the caveat of being based on just a few examples, the identification of the most preferred alternative did not depend on the hypothesis chosen. H1 produced a more clear-cut result in most instances, but not always. Since H1 and H2 reflect two different perspectives concerning the willingness to exchange an alternative for another one, our suggestion would be to use the one that better matches the perspectives of the DMs on each specific application. The possibility of using a credibility to probability cumulative distribution function for valued outranking relations adds a further layer of customization flexibility.

Further research along this line can examine adaptations of the ESS idea to other types of outranking relations in MCDA, namely those resulting from fuzzy sets theory, an aspect not addressed in this work. This work also did not address the elicitation of the *F* function for the exploitation of valued outranking relations, although the Beta function illustrated in Section 3 seems to be particularly adequate to reflect uncertainty concerning a cutting level for which the DM would be asked to indicate a minimum, a most likely, and a maximum value. On the other hand, the same example also showed that the choice of the *F* function did not qualitatively affect the conclusions obtained simply assuming the *U*(0.5, 1) distribution. Future simulation studies might inquire about whether this or some other *F* function constitutes a good rule-of-thumb to be used if the analysts do not wish to elicit it.

Appendix A

This appendix shows that the simulation process to compute the stationary probabilities, as proposed in Section 3.4, is convergent.

Lemma A.1: P^t is bounded for any *t*.

Proof: From the Chapman-Kolmogorov equation, $P_{ij}^{t+s} = \sum_{k=1}^{n} P_{ik}^{t} P_{kj}^{s}$, it is straightforward to see that, for each alternative j and each integer $t \geq 1$

$$\max_i P_{ij}^{t+1} \leq \max_l P_{lj}^{t} \text{ and } \min_i P_{ij}^{t+1} \geq \min_l P_{lj}^{t+1}.$$

Thus,

$$P_{ij}^{t+1} = \sum_{k=1}^{n} P_{ik} P_{kj}^{t} \leq \sum_{k=1}^{n} P_{ik} \max_l P_{lj}^{t} = \max_l P_{lj}^{t} \text{ and}$$

$$P_{ij}^{t+1} = \sum_{k=1}^{n} P_{ik} P_{kj}^{t} \geq \sum_{k=1}^{n} P_{ik} \min_l P_{lj}^{t} = \min_l P_{lj}^{t}.$$

This shows that, for each column j, the largest of the elements is non-increasing with t and the smallest of the elements is non-decreasing with t. Thus, the maximum and minimum elements of a column can change with t, but the range covered by those elements either shrinks or stays the same as $t \to \infty$.

It should be noted that in general P^t may not converge despite being bounded, namely when P is the transition matrix of a periodic Markov chain. For instance, for

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$\lim_{t\to\infty} Q^t$ does not exist. Furthermore, $\begin{bmatrix}\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5}\end{bmatrix} . Q^t = [0.12 \quad 0.48 \quad 0 \quad 0.16 \quad 0.24]$, if t is odd and $\begin{bmatrix}\frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5}\end{bmatrix} . Q^t = [0.08 \quad 0.32 \quad 0 \quad 0.24 \quad 0.36]$, if t is even. However, there is a unique (the algebraic multiplicity of eigenvalue $\lambda = 1$ is one) stationary distribution $\pi = \begin{bmatrix}\frac{1}{10} & \frac{2}{5} & 0 & \frac{1}{5} & \frac{3}{10}\end{bmatrix}$.

Proposition A.1. The ESS can be approximated by $[\frac{1}{n} \dots \frac{1}{n}].P^t$ for a large value of t.

Proof: This approximation is possible if P^t converges as t increases. Let us suppose that P^t does not converge. Then, the Markov chain defined by P must be either unbounded or periodic. Lemma A.1 shows it is not unbounded, therefore lack of convergence can only occur if it is periodic, as occurred for the preceding example matrix Q. Looking at the canonical form of P for a weakly (same for strongly) periodic Markov chain, which is obtained from the transition matrix of a strongly periodic Markov chain by replacing the ones by block matrices W_i (Gebali, 2008), only the last element of the first row is a non-zero element:

$$P = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & W_n \\ W_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & W_2 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & W_{n-2} & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & W_{n-1} & 0 \end{bmatrix}.$$

In our specific case, the transition matrix is defined by $p_{ij} = c_{ij}/(n-1)$ for all $i \neq j$ and $p_{ii} = 1 - \sum_{j\neq i} p_{ij}$ for all i. Thus, if the first element of the first row is zero, then all the remaining elements of that row must be non-null (more precisely all remaining elements must be $1/(n-1)$). This type of transition matrices therefore cannot correspond to the transition matrix of a periodic Markov chain. This, together with the Lemma A.1, shows by contradiction that P^t converges.

Acknowledgements

CeBER is funded by national funds, through FCT, Portuguese Science Foundation, under project UIDB/05037/2020.

References

Behzadian, M., Kazemzadeh, R. B., Albadvi, A., & Aghdasi, M. (2010). PROMETHEE: A comprehensive literature review on methodologies and applications. *European Journal of Operational Research, 200*(1), 198–215. https://doi.org/10.1016/j.ejor.2009.01.021

Belton, V., & Stewart, T. J. (2002). *Multiple Criteria Decision Analysis: An Integrated Approach*. Kluwer.

Bouyssou, D. (1992). Ranking methods based on valued preference relations: A characterization of the net flow method. *European Journal of Operational Research*, *60*(1), 61–67. https://doi.org/10.1016/0377-2217(92)90333-5

Brandt, F., Brill, M., Harrenstein, P., & Moulin, H. (2016). Tournament Solutions. In F. Brandt, V. Conitzer, U. Endriss, J. Lang, & A. D. Procaccia (Eds.), *Handbook of Computational Social Choice* (pp. 57–84). Cambridge University Press. https://doi.org/10.1017/CBO9781107446984.004

Brans, J. P., Vincke, Ph., & Mareschal, B. (1986). How to select and how to rank projects: The Promethee method. *European Journal of Operational Research*, *24*(2), 228–238. https://doi.org/10.1016/0377-2217(86)90044-5

Chung, F. (2014). A brief survey of PageRank algorithms. *IEEE Transactions on Network Science and Engineering*, *1*(1), 38–42. https://doi.org/10.1109/TNSE.2014.2380315.

Daniels, H. E. (1969). Round-robin tournament scores. *Biometrika*, *56*(2), 295–299. https://doi.org/10.1093/biomet/56.2.295

Dias, L. C., & Lamboray, C. (2010). Extensions of the prudence principle to exploit a valued outranking relation. *European Journal of Operational Research*, *201*(3), 828–837. https://doi.org/10.1016/j.ejor.2009.03.026

Dias, L. C., & Mousseau, V. (2018). Eliciting Multi-Criteria Preferences: ELECTRE Models. In L. C. Dias, A. Morton, & J. Quigley (Eds.), *Elicitation - The science and art of structuring judgement* (pp. 349–375). Springer. https://doi.org/10.1007/978-3-319-65052-4_14

Fernandez, E., Cancela, N., & Olmedo, R. (2008). Deriving a final ranking from fuzzy preferences: An approach compatible with the Principle of Correspondence. *Mathematical and Computer Modelling*, *47*(1–2), 218–234. https://doi.org/10.1016/j.mcm.2007.04.001

Fernandez, E., & Leyva, J. C. (2004). A method based on multiobjective optimization for deriving a ranking from a fuzzy preference relation. *European Journal of Operational Research*, *154*(1), 110–124. https://doi.org/10.1016/S0377-2217(02)00705-1

Figueira, J., Mousseau, V., & Roy, B. (2005). ELECTRE methods. In J. Figueira, S. Greco, & M. Ehrgott (Eds.), *Multiple criteria decision analysis: State of the art surveys* (pp. 133–153). Springer.

Franceschet, M. (2011). PageRank: Standing on the shoulders of giants. *Communications of the ACM*, *54*(6), 92–101. https://doi.org/10.1145/1953122.1953146

Gallager, R. G. (2013). *Stochastic Processes: Theory for Applications*. Cambridge University Press.

Gebali, F. (2008). *Analysis of Computer and Communication Networks*. Springer-Verlag.

Gleich, D. F. (2015). PageRank beyond the web. *Siam Review*, *57*(3), 321–363. https://doi.org/10.1137/140976649

González-Díaz, J., Hendrickx, R., & Lohmann, E. (2014). Paired comparisons analysis: an axiomatic approach to ranking methods. *Social Choice and Welfare*, *42*(1), 139–169. https://doi.org/10.1007/s00355-013-0726-2

Govindan, K., Kadziński, M., Ehling, R., & Miebs, G. (2019). Selection of a sustainable third-party reverse logistics provider based on the robustness analysis of an outranking graph kernel conducted with ELECTRE I and SMAA. *Omega*, *85*, 1–15. https://doi.org/10.1016/j.omega.2018.05.007

Greco, S., Ehrgott, M., & Figueira, J. R. (2016). *Multiple Criteria Decision Analysis - State of the Art Surveys*. Springer-Verlag.

Herrero, C., & Villar, A. (2021). Group decisions from individual rankings: The Borda–Condorcet rule. *European Journal of Operational Research*, *291*(2), 757–765. https://doi.org/10.1016/j.ejor.2020.09.043

Hokkanen, J., Lahdelma, R., Miettinen, K., & Salminen, P. (1998). Determining the implementation order of a general plan by using a multicriteria method. *Journal of Multi-Criteria Decision Analysis*, *7*(5), 273–284. https://doi.org/10.1002/(SICI)1099-1360(199809)7:5<273::AID-MCDA198>3.0.CO;2-1

Hunter, J. J. (2005). Stationary distributions and mean first passage times of perturbed Markov chains. *Linear Algebra and Its Applications*, *410*(1–3). https://doi.org/10.1016/j.laa.2005.08.005

Ishizaka, A., & Nemery, P. (2013). *Multi-criteria Decision Analysis: Methods and Software*. Wiley.

Langville, A. N., & Meyer, C. D. (2012). *Who's# 1?: the science of rating and ranking*. Princeton University Press.

Laslier, J. (1997). *Tournament solutions and majority voting*. Springer-Verlag.

Leyva López, J. C., Solano Noriega, J. J., Figueira, J. R., Liu, J., & Gastélum Chavira, D. A. (2021). Non-dominated sorting genetic-based algorithm for exploiting a large-sized fuzzy outranking relation. *European Journal of Operational Research*, *293*(2), 615–631. https://doi.org/10.1016/j.ejor.2020.12.026

Martel, J.-M., & Matarazzo, B. (2005). Other outranking approaches. In J. Figueira, S. Greco, & M. Ehrgott (Eds.), *Multiple criteria decision analysis: state of the art surveys* (pp. 198–219). Springer.

Moon, J. W., & Pullman, N. J. (1970). On Generalized Tournament Matrices. *SIAM Review*, *12*(3), 384–399. https://doi.org/10.1137/1012081

Pirlot, M. (1995). A characterization of ‘min’ as a procedure for exploiting valued preference relations and related results. *Journal of Multi-Criteria Decision Analysis*, *4*(1), 37–56. https://doi.org/10.1002/mcda.4020040104

Roy, B. (1968). Classement et choix en présence de points de vue multiples. *Revue Française d’informatique et de Recherche Opérationnelle*, *2*(8), 57–75. https://doi.org/10.1051/ro/196802V100571

Roy, B. (1991). The Outranking Approach and the Foundations of Electre Methods. *Theory and Decision*, *1*(31), 49–73.

Roy, B. (1996). *Multicriteria Methodology for Decision Aiding*. Kluwer Academic.

Roy, Bernard, & Bouyssou, D. (1993). *Aide multicritère à la décision: méthodes et cas*. Economica.

Slutzki, G., & Volij, O. (2005). Ranking participants in generalized tournaments. *International Journal of Game Theory*, *33*(2), 255–270. https://doi.org/10.1007/s00182-005-0197-5