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Two-party Bargaining Processes Based on Subjective Expectations: A Model and a Simulation Study

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Abstract

This work presents a model for a two-party bargaining process in which multiple offers are exchanged as the negotiation goes on, under a risk of breakdown. Typical game theoretical analyses of such settings assume the breakdown risk is known and the parties are able to calculate an initial offer that is immediately accepted by the other party, ending the negotiation. Aiming to develop a model that is closer to real-life situations, in which parties do exchange many offers in a bargaining process, we consider the parties are unable to compute the far-reaching consequences of their offers, and are guided by their subjective expectations of the outcome of the negotiation. This introduces a new perspective to the analysis of two-party bargaining processes: the confidence of the bargainers in terms of what they hope to achieve by bargaining with each other. The resulting model can be seen as an extension of the Zeuthen-Hicks bargaining model. We show analytically that under the assumption of concave utilities of both parties, the bargaining process converges to the nonsymmetric Nash bargaining solution, where the asymmetry is caused by differences in expectations. This result provides a new interpretation of the parameters of the nonsymmetric Nash bargaining solution, linking them to behavior in the bargaining process. As an additional contribution, we perform a simulation study to examine the role of confidence and to analyze the outcomes for non-concave utility functions.

Keywords Bargaining process · Nash bargaining solution · Negotiator confidence · Zeuthen's principle

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1 Introduction

Two-party bargaining is a classical negotiation problem with a clear practical relevance (commercial contracts, labor contracts, corporate mergers, etc.). Having been studied for a long time (Nash 1950; Harsanyi 1956), it still attracts the attention of many scholars (e.g., Bastianello and LiCalzi 2019; Dias and Vetschera 2019a; Haake and Recker 2018; Hwang 2018; Schweighofer-Kodritsch 2018). A useful abstraction of the two-party bargaining problem is to consider bargaining over one single issue. Although many real-life bargaining problems involve several issues, aggregation of multiple issues to utility values leads to a formally similar structure, if only efficient solutions are considered. The present paper therefore, following the literature, considers a single issue bilateral bargaining problem.

The literature often makes a distinction between axiomatic and strategic bargaining models (Hausken 1997; Sutton 1986). The former are mostly concerned with an axiomatic characterization of bargaining solutions, such as the well-known Nash Bargaining Solution (NBS) (Nash 1950) or the Raiffa-Kalai-Smorodinsky solution (Kalai and Smorodinsky 1975), but do not discuss the process of reaching that solution. The latter emphasize the process of bargaining and the strategic actions of parties, studying how parties decide about the offers they exchange from one round to the next. Among strategic bargaining models we can refer for instance to the classical models of Zeuthen-Hicks (Bishop 1964; Harsanyi 1956; Zeuthen 1930) and Rubinstein (1982). Strategic and axiomatic models are often related, as the Zeuthen-Hicks (Z-H) and Rubinstein models have been shown to converge to the NBS under some assumptions.

Standard game theoretical analyses of strategic bargaining models (e.g., inter alia, Muthoo 1999), are based on the concept of subgame perfect equilibria. Usually, these analyses prescribe a strategy for each party such that agreement is achieved at time zero (or approximately). One party starts the negotiation by making the equilibrium offer and this offer is immediately accepted by the other party. In a variant of this behavior, Howard (1992) proposes a game in which each party proposes one solution and in a subsequent step one of the players makes a choice involving a random event, without any real bargaining. Such strategies can be considered to be optimal for both parties in the presence of costs of haggling, such as risk of breakdown or time-discounting of the payoffs, as in Rubinstein's seminal model of bargaining (Rubinstein 1982). However, these idealized situations can hardly be considered a model of how real-world bargaining processes unfold. In this paper, we therefore consider a setting in which the parties are rational in that they maximize subjective expected utility, but are unable to compute an equilibrium a priori.

We analyze, in theory and via simulations, a model for parties that do exchange potentially many offers, one at a time. This model motivates concessions by considering the risk of a breakdown in the negotiation, which increases when parties are in a state of stalemate. Many external (and thus, from the perspective of the negotiators, random) factors can lead to a breakdown of negotiations, e.g., an attractive outside option might appear for one of the parties, the opportunity for gaining from cooperation can disappear, or a third party can intervene in the process (Muthoo 1999). In this work, we focus on subjective expectations concerning the current bargaining process, i.e., a negotiator's confidence that a good outcome will be reached negotiating with the current negotiation partner. We therefore do not consider expectations about possible favorable outside options as a source of negotiation breakdown. In many negotiations, the parties are bound to negotiate with each other (e.g., labor disputes, peace talks, consumer complaints) and thus, the possibility to strike a deal with a third party does not exist in these negotiations. Even without an outside option, there is still a risk that negotiations break down, and this risk frequently increases in a situation of stalemate. This includes, for instance, the risk that the government, a regulator, or some other authority imposes an unpleasant payoff to both parties for their inability to make progress. Another example is the case when the business opportunity might disappear (e.g. two parties negotiating a joint venture to buy a piece of land, under the risk that someone else buys it).

According to our model, the two parties (for illustrative purposes we refer to them as the seller and the buyer) make alternating offers, in which they change their position on the (single) issue under negotiation. For illustrative purposes, we refer to the issue as the price. In each step of the process, the focal party needs to choose between accepting the opponent's offer, or making a counter-offer, which could also consist of insisting on the previous offer. Note that, unless some unexpected event were to change these circumstances, quitting the negotiation is not a rational choice in our model. For example, the seller has to decide whether to accept the price offered by the buyer, or to ask for some price higher than the buyer's offer. The latter option is a gamble, as the buyer can concede and accept to pay that price, or it can insist on its own offer causing a stalemate during which the probability of a breakdown increases (an event which would result in a worse outcome). Similarly to the Z-H model (Bishop 1964; Harsanyi 1956; Zeuthen 1930), there is a critical probability of breakdown at which the seller would be indifferent between the two options. In our model, this probability depends not only on the offers on the table and their utilities for the parties, but also on their expectations about the possible final agreement.

This work therefore contributes to the literature by proposing and studying a model that focuses on a new angle for the analysis of two-party bargaining process: the confidence of the bargainers in terms of what they hope to achieve by bargaining with each other. This concept of confidence is a characteristic of each party, based on their self-confidence and their past experience, and not on the expectation that exogenous events will favour them. Assuming a few simplifications, we will show that if the two parties have concave utility functions, our model provides a full characterization of the exchange of successive offers, as well as the predicted outcome, based on a single parameter related to the negotiators' confidence. Specifically, we show that the process leads to the nonsymmetric (or asymmetric) NBS, i.e., the solution obtained in Nash's framework without the symmetry axiom (see, inter alia, Muthoo 1999; Roth 1979), in which the utilities are exponentially weighted. Therefore, although our focus is on the dynamics of the bargaining process, this work maintains a linkage between strategic and axiomatic perspectives. We also show that the exponents in the nonsymmetric NBS can be linked to the expectations of the negotiators, thus providing a new plausible source to explain the parameters of the nonsymmetric NBS.

Our analytical results hold only for concave utility functions. As a further contribution, we conduct a simulation study to examine the process and outcomes for utility functions which are not concave, as well as to assess the impact of negotiator confidence on the results, in terms of outcomes for each party, joint utility, and by how much the theoretical solution is missed.

Ensuing this introduction, Sect. 2 describes the proposed model. Section 3 provides analytical results allowing to characterize the bargaining outcome and the process of reaching that solution when the utilities are concave. Section 4 presents a simulation study for the general case in which utility functions are not necessarily concave. Section 5 presents the main conclusions and describes some of the future research paths suggested by this work.

2 Model Overview

We consider a negotiation between a buyer and a seller about one single transaction. Without loss of generality, we assume that the good to be traded has a value of zero to the seller, and a value of one to the buyer. Trading thus creates a value of one, that can be split among the two parties. The only issue to be negotiated is the price, which thus describes how the potential gain from trade is allocated to the parties. We denote the seller's offer by $s \in [0, 1]$ and the buyer's offer by $b \in [0, 1]$. The seller's utility of some price x is $u_s(x)$, and the buyer's utility is $u_b(x)$. Buyer and seller have opposing preferences: the seller prefers a higher price and the buyer a lower price. Thus we assume throughout the paper that $u'_s(x) > 0$ and $u'_b(x) < 0$.

We normalize the utility functions such that $u_s(x), u_b(x) \in [0, 1]$. Since cardinal utilities are equivalent up to a positive affine transformation, to simplify the analysis we assign a utility of zero to the disagreement outcome (denoted by d) that results from a breakdown in the negotiation. This does not imply that the outcome d is equally undesirable for both parties (e.g. the breakdown of a peace negotiation might be much more devastating for one country than for the other), as their utility scales are not comparable. In the example of a buyer-seller negotiation, the result of a breakdown would be that no transaction takes place, leaving both parties without any gain from trade. Such situations are still compatible with the chosen normalization of the utility function. All we assume here is the principle of individual rationality that no party will be worse off in any possible negotiated agreement than in the case of a breakdown. Consequently, we assume that all offers made during the process are better than the disagreement outcome d for both parties.

Let offers s and b made by the seller and the buyer, respectively, be currently on the negotiation table at a given time. Obviously, $u_s(s) > u_s(b) > u_s(d)$ and $u_b(b) > u_b(s) > u_b(d)$. Each party can choose either to accept the other party's offer, to insist on its own offer, or to adjust its own offer to a different value. Whenever both insist on their own offers (stalemate situation), the risk of breakdown increases. As mentioned in the introduction, negotiations might break down at any moment, e.g., some authority might end the negotiation and impose outcome d to punish the

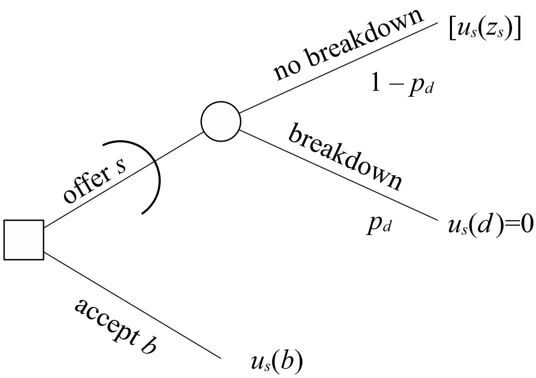


Fig. 1 Framework: Decisions in one negotiation round (Seller's perspective)

parties for not being able to make progress. However, we consider this risk to be low as long as the parties keep making successive concessions.

At a given negotiation step, the seller has the following options (the buyer's options are similar, with the necessary adjustments):

(a) Accept the buyer's offer *b*, thus ending the bargaining process;
(b) Insist on the offer *s* previously made (or, even worse, demand more), with the immediate consequence that the risk of breakdown increases;
(c) Make a concession, adjusting the current offer to a value lower than *s* (but higher than *b*) to make it more attractive to the other party, in which case the breakdown risk will decrease.

In situation a) the seller gets the certain payoff *b*, whereas in situations b) and c) the payoff remains uncertain. Even if the seller makes a concession and the breakdown risk decreases, haggling will probably continue until an agreement is reached, or the negotiation might break down at a later time. The main feature of our model is that we assume the seller (and the same applies to the buyer) cannot fully predict the final outcome unless option a) is chosen. We therefore assume the seller will make a decision based on its subjective expectations, trying at each step to choose the option that maximizes its subjective expected utility.

The situation from the seller's perspective at time *t* is depicted in Fig. 1. In this figure, the branch at the top represents all the branches corresponding to the different offers the seller could make, p_d denotes the probability that negotiations break down before an agreement is reached, and z_s denotes the certainty equivalent (expectation) that the seller subjectively estimates when considering, with its limited abilities, the future uncertainties and outcomes of the bargaining process if the negotiation does not break down. More specifically, we can write the seller's expectation as

$$z_s = u^{-1}(\gamma_s u_s(s) + (1-\gamma_s)u_s(b)) \tag{1}$$

for some $\gamma_s \in (0, 1]$. In other words, at the current stage of the negotiation process, having *b* on the table and proposing *s* as the counteroffer, the seller expects that if the negotiation does not break down, the parties will eventually settle at z_s (the subjective certainty equivalent of continuation), which for the seller has an utility $u_s(z_s)$. This utility must be in-between $u_s(b)$ and $u_s(s)$. Constraining γ_s to the interval (0, 1] is reasonable, because as long as the seller does not accept the buyer's offer, continuation of the negotiation implies that $z_s > b$, and since the buyer is not going to offer anything better than the seller's offer, we also have $z_s \leq s$. Since the seller's utility function is monotonic, the utility of any outcome that lies between b and s can be written as a linear combination of $u_s(b)$ and $u_s(s)$.

From the perspective of a seller maximizing expected utility, continuing the negotiation is preferable to accepting b iff:

$$\text{offer } s \succ \text{ accept } b \Leftrightarrow (1 - p_d)u_s(z_s) > u_s(b) \tag{2}$$

Thus, the seller will choose to offer s if the breakdown probability p_d fulfills the condition

$$p_d < \frac{u_s(z_s) - u_s(b)}{u_s(z_s)} \tag{3}$$

Analogously, from the perspective of the buyer,

$$\text{offer } b \succ \text{ accept } s \Leftrightarrow (1 - p_d)u_b(z_b) > u_b(s) \tag{4}$$

which results in the condition

$$p_d < \frac{u_b(z_b) - u_b(s)}{u_b(z_b)} \tag{5}$$

We assume that each party knows its own preferences and expectations, but not the other party's. Per Eqs. (3) and (5), we can conclude that if p_d is initially low then both parties will insist on their offers. This causes p_d to increase at each time step. At a given time step, one of the inequalities, (3) or (5), will no longer hold. For instance, consider that at time t the seller observes inequality (3) no longer holds, and therefore $b \succeq s$. It is therefore not anymore rational to hold on to offer $s^{(t)} = s$ at time t (we assume no bluffing from either party). However, this does not mean the seller must accept b, it can also adjust its offer to a new value $s^{(t+1)} < s^{(t)}$. The seller will prefer making this concession to accepting b as long as the expected value $(1 - p_d)u_s(z_s)$ is above $u_s(b)$.

A similar argument applies to the buyer, who will also observe the necessity to make a concession as soon as condition (5) is violated. Comparing the two conditions, the seller will make a concession before the buyer does if $\frac{u_s(z_s)-u_s(b)}{u_s(z_s)} < \frac{u_b(z_b)-u_b(s)}{u_b(z_b)}$, i.e.,

$$u_s(z_s)u_b(s) < u_s(b)u_b(z_b), \tag{6}$$

and the buyer would make a concession before the seller if the reverse condition holds.

Then, a new cycle of increasing breakdown probability would be reiterated until again one of the parties needs to concede. Following this simple process, as both parties keep making concessions successively enforcing and then reversing inequality (6), they will eventually reach an agreement (except if the breakdown event occurs despite its low probability).

The probability p_d influences the decision of a party whether to make a concession or not. In actual negotiations, the two parties will only have subjective estimates of p_d, which might not be correct and which could be different between the two parties. Although we do not explicitly consider errors in estimating p_d in our model, we note that our basic line of reasoning will also apply if the parties use incorrect estimates of p_d. As long as the parties are able to note an increase in the objective p_d caused by a stalemate, such an increase will cause them at some point in time to make a concession, even if they do not know the true value of p_d exactly. If both parties overestimate p_d, then they will both make concessions sooner than needed and will reach an agreement more quickly. If both parties underestimate p_d, then they will both prolong the stalemate under an increasing risk they do not perceive, and negotiations are more likely to break down. If one party overestimates p_d while the other party underestimates it, the overestimating party will be at a disadvantage making concessions faster than needed, but the other party's misjudgement makes it more likely that negotiations break down.

As long as the parties' estimates are biased in a similar way, the seller will make a concession before the buyer if condition (6) is fulfilled, otherwise the buyer. The same condition could also be obtained by applying the standard reasoning of the Z-H model, which assumes that both parties are able to estimate not only their own but also their opponent's critical probability of a breakdown and that the party who has a lower critical probability makes a concession. Although we consider our arguments for this condition to rely on weaker assumptions about the parties' information and cognitive abilities, we note that the analysis will be the same regardless of how condition (6) is obtained. Thus, our model of two rational but "shortsighted" parties also reflects Zeuthen's principle, which according to Harsanyi (1977) is the only rule consistent with subjective probabilities that rational players can entertain about each other's behavior: the next concession will come from the party less willing to risk a conflict.

The standard Z-H model (Bishop 1964; Harsanyi 1956) considers a simpler setting, in which the opponent can react to an offer only by accepting it or terminating the negotiation (leading to the disagreement outcome with utility zero). A rational seller, who estimates the probability of the buyer to reject the seller's offer and terminate the negotiation to be p_b will make an offer s if

$$p_b u_s(d) + (1 - p_b) u_s(s) > u_s(b) \tag{7}$$

From this inequality, the critical probability p_b, which would make the seller indifferent between the two options, is given by

$$p_b = \frac{u_s(s) - u_s(b)}{u_s(s) - u_s(d)} \tag{8}$$

Similarly, the critical probability from the buyer's perspective (the probability that the seller will reject the buyer's offer) is

$$p_s = \frac{u_b(b) - u_b(s)}{u_b(b) - u_b(d)} \tag{9}$$

Zeuthen's principle asserts that the seller wants to achieve

$$p_b > p_s \tag{10}$$

By substituting (8) and (9) and taking into account that $u_s(d) = u_b(d) = 0$, condition (10) becomes

$$u_b(s)u_s(s) > u_b(b)u_s(b) \tag{11}$$

If the seller has managed to establish condition (11), the buyer will then try to reverse it. Therefore, each side wants to maximize the product of utilities of its offer, so the process will converge to the NBS. If the utility functions of both sides are concave, it can be shown (Dias and Vetschera 2019a) that the function $u_b(x)u_s(x)$ is quasiconcave and therefore has a single maximum. Otherwise, if parties use only local information, they might get stuck in a local maximum and fail to reach the NBS.

The Z-H model coincides with a specific case of our model in which $z_s = s$ and $z_b = b$. Thus, both parties expect to get exactly what they are asking for if the negotiation does not break down. Our model assumes a more realistic (lower) expectation z_s. A seller who is very confident (has a high γ_s in eq. (1)) will assume that the negotiation will end somewhere near its own offer. In contrast, a less confident seller will assume the opposite. This notion of confidence is inherent to the seller and refers to its expectations concerning the final outcome of the negotiation with this specific buyer, and therefore it is different from other expectations not considered in this model, such as the hope that an attractive outside offer might appear.

By introducing this notion of confidence as an inherent characteristic of the negotiator, we add a descriptive element to the model. This extension towards a more behavioral model is in line with empirical literature utilizing the Z-H model (such as Svejnar 1986), which explains asymmetry in bargaining outcomes by differences in individual characteristics of the parties such as risk attitudes. Our model also explicitly acknowledges that each party has many options besides insisting on its own offer or accepting the other party's offer, namely the model includes all the options corresponding to making a different offer. In addition, it provides a plausible interpretation of Zeuthen's principle in terms a breakdown risk successively increasing during a stalemate.

3 Model Analysis

This section characterizes the bargaining process dynamics according to the extended model presented in the previous section. First, we show that these dynamics depend essentially on the confidence parameters of both parties. Then, we characterize the path of the successive exchange of offers, with alternating reversals of inequality (6). Finally, we provide a characterization of the solution to which this path will lead.

As mentioned in the previous section, in the particular case that $\gamma_s = \gamma_b = 1$, the model becomes a standard Z-H model that leads to the NBS. We first note that the NBS can still be obtained even if the two parties are not fully confident, as the following proposition shows:

Proposition 1 *The bargaining process will converge to the (symmetric) NBS if the confidence levels of both parties are equal and strictly positive.*

Proof Substituting the definitions of $u_s(z_s)$ and $u_b(z_b)$ into (6) yields the condition

$$[\gamma_s u_s(s) + (1-\gamma_s)u_s(b)]u_b(s) > u_s(b)[\gamma_b u_b(b) + (1-\gamma_b)u_b(s)] \tag{12}$$

We denote the common level of confidence by $\gamma_s = \gamma_b = \gamma$. Condition (12) then becomes

$$[\gamma u_s(s) + (1-\gamma)u_s(b)]u_b(s) > u_s(b)[\gamma u_b(b) + (1-\gamma)u_b(s)] \tag{13}$$

Since the term $(1-\gamma)u_s(b)u_b(s)$ cancels out and $\gamma > 0$, this is equivalent to (11). □

Proposition 1 already provides a hint that the outcome of the process depends on the relative magnitude of the two confidence parameters rather than on their values. The following proposition shows that this is indeed the case:

Proposition 2 *If the confidence parameters of both parties are strictly positive, the outcome depends only on the ratio of confidence parameters, not on their absolute values.*

Proof Let $r = \gamma_s/\gamma_b$ denote the ratio of the two confidence parameters. Then inequality (12) can be written as

$$[r\gamma_b u_s(s) + (1-r\gamma_b)u_s(b)]u_b(s) > [\gamma_b u_b(b) + (1-\gamma_b)u_b(s)]u_s(b) \tag{14}$$

$$\Leftrightarrow r\gamma_b u_s(s)u_b(s) + u_s(b)u_b(s) - r\gamma_b u_s(b)u_b(s) > \gamma_b u_s(b)u_b(b) + u_s(b)u_b(s) - \gamma_b u_b(s)u_s(b) \tag{15}$$

$$\Leftrightarrow r\gamma_b u_s(s)u_b(s) - r\gamma_b u_s(b)u_b(s) > \gamma_b u_s(b)u_b(b) - \gamma_b u_b(s)u_s(b) \tag{16}$$

Dividing this inequality by $\gamma_b > 0$ yields the equivalent condition which depends only on r:

$$r(u_s(s) - u_s(b))u_b(s) > (u_b(b) - u_b(s))u_s(b) \tag{17}$$

□

Note that as long as the seller's offer is better for the seller than the buyer's offer, $u_s(s) - u_s(b) > 0$. Thus, the larger r, the easier it is to establish that inequality. This means a seller who is more confident can afford to make smaller concessions (and end up at a better value). As the difference $s - b$ (and consequently because of monotonicity also $u_s(s) - u_s(b)$ and $u_b(b) - u_b(s)$) decreases, both sides of the above inequality tend to zero.

For the remainder of this section, we assume that both utilities are concave. As the simulation results in the following section will show, this assumption is crucial for many properties of the model, and no clear predictions about the outcome of the process can be made for non-concave utilities.

For the following analysis, we define a function $f(b, s)$ as the difference between the left hand and right hand side of (17):

$$f(b, s) = r(u_s(s) - u_s(b))u_b(s) - (u_b(b) - u_b(s))u_s(b) \tag{18}$$

As long as $f(b, s) < 0$, the seller will change s to increase its value above zero, and if $f(b, s) > 0$, the buyer will change b to decrease f below zero. We first analyze whether these changes actually correspond to concessions of the respective parties. A concession by the seller means lowering the price s demanded by the seller, a concession by the buyer means increasing b. Thus, the seller has an incentive to make a concession (decrease s in order to increase f) if

$$\frac{\partial f}{\partial s} < 0 \tag{19}$$

and the buyer has an incentive to make a concession (increase b in order to decrease f) if

$$\frac{\partial f}{\partial b} < 0 \tag{20}$$

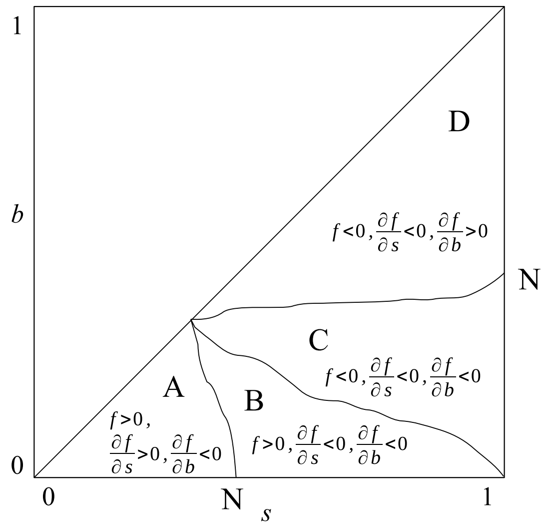
For our analysis, we represent the problem in b/s space and analyze in which regions of that space the above conditions are fulfilled. Note that as soon as $s \leq b$, both sides would accept the other side's offer, so we have to consider only situations in which $s > b$. Furthermore, $f(b, s) = 0$ for $s = b$. We now show that for $r < 1$, the area defined by $\{(b, s) : 0 \leq b \leq 1, 0 \leq s \leq 1, b < s\}$ can be partitioned as shown in Fig. 2.

In the regions labeled A and B, the buyer needs to make a concession, as $f(b, s) > 0$. Since in both regions $\partial f / \partial b < 0$, the buyer here needs to increase the offered price in order to decrease f. Similarly, the seller will make a concession and decrease the demanded price in regions C and D.

The partitioning shown in Fig. 2 is characterized by the following properties:

- The curve separating regions A and B, i.e., the curve along which $\partial f / \partial s = 0$, is monotonically decreasing and intersects with the line $b = 0$ at $s = N$, where N is the price in the NBS.
- The curve separating regions B and C, i.e., the curve along which $f = 0$, passes through the point $(b = 0, s = 1)$ and is monotonically decreasing.
- The curve separating regions C and D, i.e., the curve along which $\partial f / \partial b = 0$, is monotonically increasing and intersects with the line $s = 1$ at $b = N$, where N is the price in the NBS.

Fig. 2 Partitioning of *b*/*s* space when the utilities are concave and $r < 1$



– All three curves intersect with the line $s = b$ at the same point.

Similarly, if $r > 1$, the partitioning is characterized by the following properties:

– The curve separating regions A and B, i.e. the curve along which $\partial f/\partial s = 0$, is monotonically increasing and intersects with the line $b = 0$ at $s = N$, where N is the price in the NBS.
– The curve separating regions B and C, i.e. the curve along which $f = 0$, passes through the point $(b = 0, s = 1)$ and is monotonically decreasing
– The curve separating regions C and D, i.e. the curve along which $\partial f/\partial b = 0$, is monotonically decreasing and intersects with the line $s = 1$ at $b = N$, where N is the price in the NBS.
– All three curves intersect with the line $s = b$ at the same point.

Proofs of these properties are provided in the appendix. Obviously, the fact that the lines at which partial derivatives are zero intersect with the boundaries at the NBS, and their monotonicity together imply that for $r > 1$ (the seller is more confident than the buyer), the three lines will intersect at a price which is larger than the Nash outcome (i.e., better than the NBS for the seller), while for $r < 1$ (the buyer is more confident), they will intersect at a price lower than the Nash outcome (i.e., better for the buyer).

This partitioning now allows us to make predictions about the bargaining process. One of the parties will start the process by making an offer that is more attractive than d for the other party (and also to itself, of course). Considering the parties are initially ambitious, the initial offer will be close to the extreme positions (i.e., the buyer offering zero, the seller demanding one), at the bottom right corner in Fig. 2, i.e., the initial situation will be a point in region B or C. If an initial offer from the buyer places the process in region C, the seller has an incentive to make a concession to move from region C to B. Similarly, if an offer from the seller places the process in region B, the buyer then has an incentive to move from region B to C. The process thus consists of alternating concessions so that offers oscillate between regions B and C around the curve $f = 0$. Ultimately, the process moves toward the point at which all three curves intersect. Here, $s = b$, thus, the two parties agree on the price.

The point in which all three curves intersect has several properties. The fact that both derivatives are zero at a point in which the offers of both sides are identical implies that this outcome is stable, since no party has a incentive to deviate from this point. We define

Definition 1 A locally stable agreement (LSA) is a value x so that for $s = x$: $\partial f(x, s)/\partial s = 0$ and for $b = x$: $\partial f(b, x)/\partial b = 0$.

This definition of an LSA involves changes of offers in both directions. In fact, once a point on the line $s = b$ is reached, it would not be rational for the parties to make any further concession, since the opponent would be willing to accept the currently stated price. This definition of an LSA thus also allows considering whether parties would have an incentive to retract a concession and toughen their position (i.e., whether the buyer has an incentive to ask for a lower and the seller to ask for a higher price). Although such behavior would violate the principle of bargaining in good faith, we still consider the absence of incentives for such behavior an important characteristic of a stable bargaining outcome. Indeed, from the definition of f in (18), we derive:

$$\partial f(b, s)/\partial s = ru'_s(s)u_b(s) + u'_b(s)u_s(b) \tag{21}$$

$$\partial f(b, s)/\partial b = -ru_b(s)u'_s(b) - u_s(b)u'_b(b) \tag{22}$$

Thus, if $s = b = x$, the derivatives have opposite sign. If x is not an LSA, then either $\partial f(x, s)/\partial s|_{s=x} > 0$, which means the seller has an incentive to ask for a higher price, or $\partial f(b, x)/\partial b|_{b=x} > 0$, which means the buyer has an incentive to ask for a lower price.

As we show in Proposition 8 in the appendix, the curves where $\partial f/\partial s = 0$ and $\partial f/\partial b = 0$ intersect at a point where $s = b$, therefore that point is an LSA. We now provide a characterization of an LSA in terms of the agreement value:

Proposition 3 *An agreement $s = b = x$ is a locally stable agreement if and only if x is a local extremum of $W(x) = u_s(x)^r u_b(x)$ (or equivalently $W(x) = u_s(x)^{\gamma_s} u_b(x)^{\gamma_b}$ or $W(x) = u_s(x)u_b(x)^{1/r}$)*

Proof Since the power is a monotonic function, the three functions stated in Proposition 3 obviously take their maximum or minimum at the same values of x. We therefore consider function $u_s(x)^r u_b(x)$.

An LSA implies that $\partial f(b, s)/\partial s = 0$. From the definition of f in (18), we obtain

$$\partial f(b,s)/\partial s = ru_s'(s)u_b(s) + u_b'(s)u_s(b) = 0 \tag{23}$$

At the intersection with $s = b$, therefore

$$ru_s'(s)u_b(s) + u_b'(s)u_s(s) = 0 \tag{24}$$

must hold. Similarly

$$\partial f(b,s)/\partial b = -ru_b(s)u_s'(b) - u_s(b)u_b'(b) = 0 \tag{25}$$

and at the intersection with $s = b$.

$$-ru_s'(s)u_b(s) - u_b'(s)u_s(s) = 0 \tag{26}$$

At an extremum of $W(x)$, the first order condition $W'(x) = 0$ must hold.

$$W_r'(x) = ru_s'(x)u_s(x)^{r-1}u_b(x) + u_s(x)^r u_b'(x) = 0 \tag{27}$$

Dividing by $u_s(x)^{r-1} > 0$ gives (24) as well as (26), so the two conditions are equivalent. □

Note that Proposition 3 does not require the assumption that both utility functions are concave. However, this assumption is needed to demonstrate that this LSA is unique.

Proposition 4 *If the utility functions of both parties are concave, there exists only one LSA, which corresponds to the unique maximum of $W(x)$, denoted x^*. Moreover no party has incentives to unilaterally deviate from x^*, i.e., $f(x^*, s) < 0, \forall s > x^*$ and $f(b, x^*) > 0, \forall b < x^*$.*

Proof Here we consider the function $u_s(x)^r u_b(x)$ for $r \leq 1$, the proof for $r \geq 1$ using $u_s(x)u_b(x)^{1/r}$ is analogous. First, we show there exists a unique LSA. The second derivative of W is:

$$\begin{aligned} W_r''(x) = {} & r(r-1)u_s(x)^{r-2}u_s'(x)^2 u_b(x) \\ & + ru_s(x)^{r-1}u_s''(x)u_b(x) \\ & + 2[ru_s(x)^{r-1}u_s'(x)u_b'(x)] \\ & + u_s(x)^r u_b''(x) \end{aligned}$$

Since $r \leq 1$ and all utilities and $u_s'(x)$ are positive, the first term is not positive. The second and fourth term are negative by concavity of the utility functions and the third term is negative because $u_b'(x) < 0$. Thus, at least three terms are negative and none is positive. This means that $W(x)$ is a concave function, which has a single maximum at the point x where $W'(x) = 0$. In Proposition 3, we have already shown that for any utility functions, this first order condition is fulfilled at an LSA, and therefore this unique maximum corresponds to an LSA. Furthermore, we have shown in Proposition 8 in the appendix that only one point can exist at which both derivatives $\partial f/\partial s$ and $\partial f/\partial b$ are zero. So the LSA at this maximum is also the only LSA that exists for concave utilities.

Let us now show that no party has an incentive to revert this agreement. We show (in the Appendix) in Lemma 4 that for $b < s$, $\partial^2 f/\partial^2 s < 0$ and in Lemma 3 that $\partial^2 f/\partial^2 b > 0$. The value $s = x^*$ is a maximum for $f(x^*, s)$ as a function of s, since $\partial f(b,s)/\partial s|_{b=s=x^*} = 0$ and $\partial^2 f(b,s)/\partial s^2 < 0$, and therefore $f(x^*, s)$ is less than $f(x^*, x^*) = 0$ for $s > x^*$. Similarly, the value $b = x^*$ is a minimum for $f(b, x^*)$ as a function of b , since $\partial f(b,s)/\partial b|_{b=s=x^*} = 0$ and $\partial^2 f(b,s)/\partial b^2 > 0$, and therefore $f(b, x^*)$ is greater than $f(x^*, x^*) = 0$ for $b < x^*$. □

Thus, if the parties' utilities are concave, the bargaining process converges to this unique LSA, which is the nonsymmetric NBS $\operatorname{argmax}_x u_s(x)^{\gamma_s} u_b(x)^{\gamma_b}$, which coincides with $\operatorname{argmax}_x u_s(x)^r u_b(x)$ as $\gamma_b > 0$.

If at least one of the two utility functions is not concave, $W(x)$ is no longer necessarily concave and thus it might have multiple local maxima and minima. Proposition 3 implies that the two parties do not have an incentive to move away from a local minimum, so a local minimum would also be an LSA. However, we note that from a point that is neither a local maximum nor a local minimum, it is more likely that the parties move towards the local maximum of $W(x)$ rather than towards the local minimum.

Consider a local maximum $x^{\max}$ and the two neighboring local minima below and above this maximum, denoted $x^{\min_1}$ and $x^{\min_2}$, so that $x^{\min_1} < x^{\max} < x^{\min_2}$ and the interval $(x^{\min_1}, x^{\min_2})$ does not contain any other extrema of $W(x)$. Let x^a be some arbitrary point $x^{\max} < x^a < x^{\min_2}$. Since x^a is located above the maximum of $W(x)$, $W'(x^a) < 0$, which implies $\partial f/\partial b > 0$. Therefore this solution would be unattractive to the buyer, who would be overpaying as $f(b, x^a)$ would be negative for $b < x^a$ (the buyer would have an incentive to retract from a possible agreement $s = b = x^a$ and ask for a lower price).[1] Now suppose that x^a is such that $x^{\min_1} < x^a < x^{\max}$. Since x^a is located below the maximum of $W(x)$, $W'(x^a) > 0$, which implies $\partial f/\partial s > 0$. Therefore this solution would be unattractive to the seller, who would be conceding too much as $f(x^a, s)$ would be positive for $s > x^a$ (the seller would have an incentive to retract from a possible agreement $s = b = x^a$ and ask for a higher price).[2]

In summary, these results provide a characterization of the negotiation process and the final outcome when the buyer's and the seller's utility functions are concave. If the buyer and the seller are equally confident ($r = 1$) then the process converges

[1] Suppose the buyer asks for price $x^a - \Delta$. This creates a situation in which $f(x^a - \Delta, x^a) < 0$, so now the seller needs to move away from x^a, too. Obviously, selecting $s = x^a - \Delta$ would bring the negotiation to $f(x^a - \Delta, x^a - \Delta) = 0 > f(x^a - \Delta, x^a)$. However, it is not clear whether a seller acting only on local information would make that choice. For $r > 1$, the seller will necessarily have to make a regular concession. From Lemma 5 in the appendix (which does not depend on the utility functions being concave), we know that for $r > 1$, $\partial^2 f/\partial s \partial b > 0$. Since $\partial f/\partial s$ is already negative at $s = b = x^a$, decreasing b will further decrease it. Therefore, the seller then will locally increase f by reducing s. However, for $r < 1$, it is not clear whether the seller will have a local incentive to make a concession, or to make a reverse concession.

[2] In this case, Lemma 5 in the appendix allows saying that the buyer would need to accept the price increase if $r < 1$, but this would not be guaranteed if $r > 1$.

to the well-known NBS. Otherwise, it converges to the nonsymmetric NBS, placing more weight on the utility of the more confident party. However, these results do not allow reaching conclusions about the outcome of the negotiation if the utilities are not concave, where the process might e.g. lead to a local minimum of W. This motivates the simulation study presented in the following section.

4 Simulation

We use a simulation analysis to explore the properties of the model for non-concave utility functions. Specifically, we want to study whether the process still converges to a local maximum of the nonsymmetric Nash function identified in Proposition 3, or how often it fails to do so. Furthermore, we use the simulation to quantify the effects of the confidence parameters on the individual outcomes of the parties, and on other properties of the agreement. The analysis of the preceding section has shown that in the agreement, the party having the higher level of confidence will be better off than in the (symmetric) NBS. In the simulation study, we can quantify this effect and verify whether it will also hold when the utility functions are not concave. Furthermore, we can study how far the solutions deviate from the NBS, and to quantify the possible loss in efficiency (joint utility) that results from this deviation.

For the simulation, arbitrary pairs of monotonic (but not necessarily concave) utility functions were generated using the bisection approach of Dias and Vetschera (2019b). Utility functions for both sides are represented as utility values for equally spaced prices in the zero-one interval. Since the method is based on a bisection approach, it works most efficiently if the number of intervals is a power of two. For the present simulation, $2^9 = 512$ intervals were used.

Each pair of utility functions defines a problem. In total, two sets of 100,000 problems each were generated using different random number streams to check the stability of results. For each problem, the two confidence parameters γ_s and γ_b were varied from 0.05 to 1 in steps of 0.05, thus generating 400 combinations of confidence parameters. Since Proposition 2 holds for arbitrary (and not just for concave) utility functions, some of these combinations should generate identical outcomes. They were nevertheless all included in the simulation to test the numerical stability of the simulation, and the results confirmed that.

The bargaining process was simulated in the following way: The initial offer of one party was set to the best outcome for that party (1 for the seller or 0 for the buyer), and the initial offer of the other party was set to that party's second best outcome ($1 - 1/512$ for the seller or 1/512 for the buyer). Between problems, the party starting with the second best outcome was alternated. Given the two offers, the value of f was calculated to decide which party needs to make a concession. If $f < 0$, the seller has to make a concession, otherwise, the buyer. The conceding party makes a concession by moving to the first discrete price level that would revert the sign of f (i. e., the seller moves to the highest price smaller than its current offer that leads to $f > 0$, the buyer to the lowest price above its previous offer that would lead to $f < 0$). Denote the current offer of the buyer and seller by s_t and b_t and the set of possible prices by $X = \{x_i\} = \{0, 1/512, 2/512, ...1\}$. The next offer s_{t+1} of the seller is then given by

$$s_{t+1} = \max_{i: x_i < s_t \wedge f(b_t, x_i) > 0} x_i \tag{28}$$

Concessions of the buyer are determined in an analogous way. The process terminates when

$$s_{t+1} \leq b_t \vee b_{t+1} \geq s_t \tag{29}$$

i.e., when one party cannot do better than accept the offer from the other party. Since each step moves the offer of one side towards the offer of the other side, and there is only a finite number of possible offers, the process will always converge to a solution.

Note that this convergence property relies on the assumption that both parties bargain in good faith, i.e., they do not retract from a previously made offer. For non-concave utilities, it could be the case that e.g. for a given offer b_t of the buyer, there exists a value $s' > s_t$ so that $f(b_t, s') > 0$. Thus the seller could achieve $f > 0$ by increasing, rather than decreasing, the price he or she demands. The assumption of bargaining in good faith excludes such moves, which could cause the process to oscillate infinitely between some offers.

As Fig. 3 shows, the partitioning of *b*/*s* space into compact subspaces, that existed for concave utilities, no longer exists if utilities are not concave. In Fig. 3, regions in which $f > 0$ (corresponding to regions A and B in Fig. 2) are marked in dark gray, regions in which $f < 0$ (corresponding to C and D) in light gray. Within these regions, the parts in which the partial derivatives of f with respect to s and b are positive and negative are also scattered (we did not depict these regions in the figure to reduce its complexity). Hollow circles along the line $s = b$ mark local maxima of the nonsymmetric Nash function $u_s(x)^r u_b(x)$, the solid circle marks the global maximum. Figure 3 shows a problem in which the process converged to the global maximum for all values of r. Clearly, the maximum depends on r, the price in the agreement increases for a higher r, i.e., the more confident the seller is. In the two examples with $r < 1$, reaching the agreement in some cases required quite large concessions from the seller to move across a region in which $f < 0$ (the long black line across the light gray region), for example, in the negotiation with $r = 1/5$, the seller had to decrease the price from about 0.76 to 0.37 in one step. Similarly, in the negotiations represented in the lower part of Fig. 3, the buyer in one step had to make a large concession (across the dark gray region).

However, the process does not always converge to the global and sometimes not even to a local maximum of the nonsymmetric Nash function, as the example in Fig. 4 shows. Here, the global maximum is located at approximately $s = b = 0.12$, but the process converged to $s = b = 0.44$. The reason for this deviation is also quite obvious from the figure: The large region around $s = 0.8$ in which $f > 0$ (marked in dark gray) forces the buyer to increase its offer up to the level of 0.44, while the seller has to make only small concessions. Once that level is reached, however, there is no more possibility for the seller to achieve $f > 0$ at a price higher than the buyer's offer, so the seller accepts the offer from the buyer.

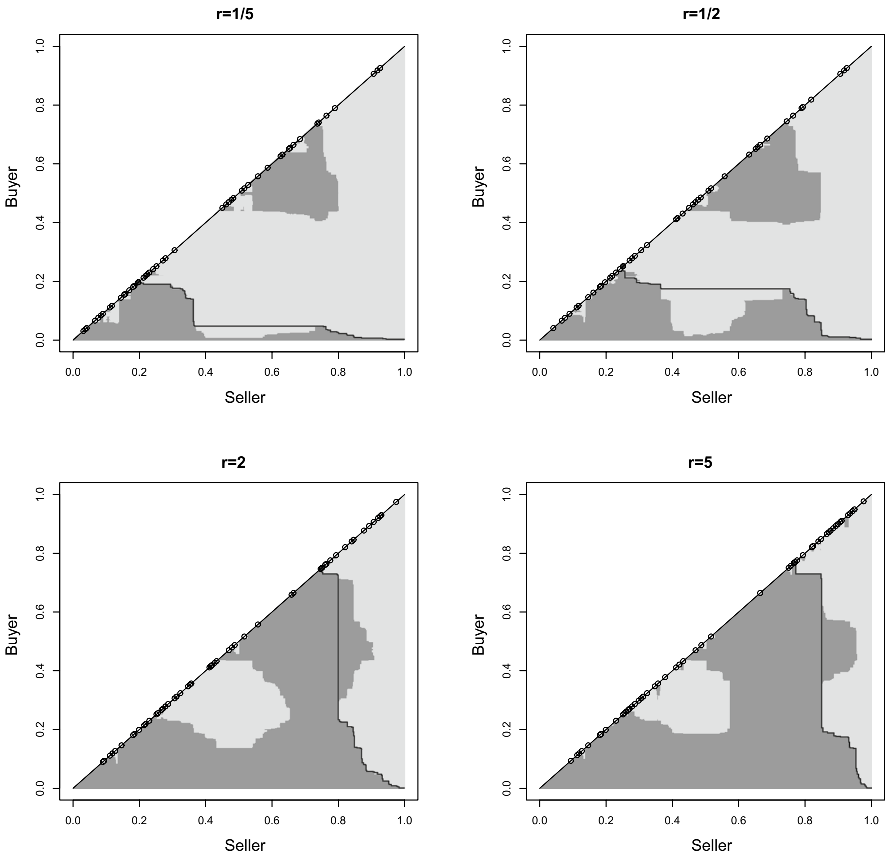


Fig. 3 Examples of partitioning of the *b*/*s* space and negotiation path for non-convex utilities and different values of *r*

Fig. 4 Example of a negotiation not converging to a maximum of the nonsymmetric Nash function

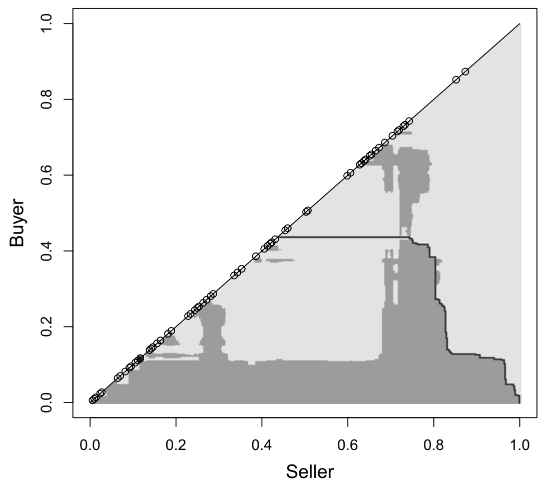


Table 1 Fraction (in %) of cases in which the process did not converge to the global or to a local maximum of the nonsymmetric Nash function

γ_s	γ_b							
	0.05		0.35		0.65		0.95	
	Global	Local	Global	Local	Global	Local	Global	Local
0.05	0.00	0.00	27.45	0.43	33.92	0.92	37.65	1.57
0.35	27.81	0.46	0.00	0.00	9.45	0.08	14.90	0.04
0.65	33.17	0.84	9.73	0.07	0.00	0.00	6.01	0.01
0.95	36.99	1.32	14.74	0.05	5.56	0.06	0.00	0.00

Table 1 shows the fraction of cases in which the process did not converge to a local or the global maximum of the nonsymmetric Nash function. Here we present only a few selected values of the two confidence parameters, the intermediate values do not offer much additional insight and can be obtained from the authors upon request.

Given the high number of local optima shown in Figs. 3 and 4, it is not surprising that the global maximum was not reached in a considerable number of cases. As we have already shown, if the confidence parameters of the two parties are equal, the process will always converge to the NBS (which in that case is also equal to the nonsymmetric NBS), so in these cases, all entries are zero. However, it should be noted that the simulation assumes that both parties have full information about both utility functions and therefore will make large concessions if such concessions are needed to reach the global maximum. Otherwise, the existence of local maxima could prevent them from converging to an agreement. This situation is similar to the standard Z-H model, where local maxima could also prevent convergence to the NBS (Dias and Vetschera 2019a). If the two confidence parameters are more imbalanced, the process fails more often. The highest value is reached when one party is highly confident (0.95), and the other has extremely low confidence (0.05). The situation of Fig. 4, in which the process also fails to reach a local maximum, is quite rare. Even in very unbalanced settings, it occurs in only about 1.5% of all cases.

To study whether failing to find the global maximum actually implies a large loss in performance, we compare the value of the nonsymmetric Nash function that was achieved in the agreement to its global maximum.

As Table 2 shows,[3] the agreement reached achieves well over 80% of the global optimum except when the difference between the two parties is extreme. If both sides have a confidence parameter of more than 0.35, the process converges to a solution that provides more than 95% of the global maximum and thus in effect performs almost as well as the global optimum of the nonsymmetric NBS.

[3] The value of the global maximum is not the same for $r = 1/2$ and $r = 2$. Thus, to compute the loss in performance in a balanced way, Table 2 uses the function $u_s(x)^{\gamma_s/\gamma_b} u_b(x)$ for $\gamma_s > \gamma_b$ and $u_s(x) u_b(x)^{\gamma_b/\gamma_s}$ for $\gamma_s < \gamma_b$.

Table 2 Ratio (in %) of the nonsymmetric Nash function achieved in the agreement to the value in the global optimum

γ_s	γ_b			
	0.05	0.35	0.65	0.95
0.05	100.0	88.0	82.0	78.2
0.35	87.5	100.0	98.4	96.2
0.65	82.1	98.5	100.0	99.4
0.95	78.8	96.0	99.4	100.0

Table 3 Performance of the seller relative to the seller's outcome in the NBS (in %) for different levels of the confidence parameters

γ_s		γ_b			
		0.05	0.35	0.65	0.95
0.05	min	100.0	25.1	14.4	10.1
	avg	100.0	60.6	53.0	48.9
	max	100.0	100.0	100.0	100.0
0.35	min	100.0	100.0	69.9	53.8
	avg	128.9	100.0	84.1	75.3
	max	175.0	100.0	100.0	100.0
0.65	min	100.0	100.0	100.0	81.0
	avg	131.5	114.5	100.0	90.1
	max	185.5	130.2	100.0	100.0
0.95	min	100.0	100.0	100.0	100.0
	avg	132.5	121.0	109.4	100.0
	max	189.8	146.2	119.1	100.0

As a final analysis, we now study the effect of differences in the confidence values on individual and collective outcomes.

Table 3 shows that the party having the lower confidence parameter will never be able to perform better than in the (symmetric) NBS, while the party having the higher confidence parameter will always achieve at least the outcome it would obtain in the NBS. The performance effect of being the weaker party is quite strong, both on average and in the worst case. Conversely, a party that is much stronger can achieve almost twice the outcome it would obtain in the NBS.

The overall efficiency of the outcome is not affected very strongly by differences in the confidence parameters. Table 4 shows the joint utility (sum of utilities of the two parties) obtained in the agreement as a fraction of the maximum joint utility that could have been obtained in any agreement. Since the NBS does not maximize the sum of the two utilities, the main diagonal of that table (where the process always converges to the NBS) is below 100%.

Table 4 Efficiency (joint utility) of the agreement (in % of the possible maximum), for different combinations of confidence parameters

γ_s	γ_b			
	0.05	0.35	0.65	0.95
0.05	99.7	94.1	91.5	90.0
0.35	94.1	99.7	98.9	97.7
0.65	91.5	98.9	99.7	99.4
0.95	90.0	97.7	99.4	99.7

5 Conclusions

In this paper we studied, analytically and through simulations, a model for the exchange of multiple offers among two bargaining parties facing a breakdown risk. The model is characterized by the utility functions of the parties and parameters (confidence) related to their subjective expectations of the outcome.

The analysis of the model provides a full characterization of the exchange of successive offers, as well as the predicted outcome, if the two parties have concave utility functions. In this case, the final outcome will be a nonsymmetric NBS, in which the utility functions of both parties are exponentially weighted by their confidence parameters, or equivalently, the utility of one party is weighted by the ratio of the confidence parameters. Consequently, the process converges to the symmetric NBS if the confidence parameters of both parties are equal. The analysis also shows the well-known Z-H model can be included as a particular case in which the parties are maximally confident. The confidence parameters can be seen as a descriptive element, able to represent the idea observed in practice that negotiator characteristics such as their self-confidence or their optimism can influence the outcome of a negotiation. The finding that it is the relative magnitude that matters, not the absolute value, and the analysis of the resulting imbalance provides the interesting insight that two highly confident parties would reach the same result as two parties with low confidence, if the confidence ratio is the same. We therefore connect this behavior to a single parameter that can be estimated, and can provide a new framework for empirical research.

In addition, this analysis contributes a new possible explanation for obtaining the nonsymmetric NBS. Kalai (1977) has shown that the nonsymmetric NBS can be obtained by n-person replications of Nash's original setting, considering that two parties represent two groups of different size. Other paths to obtain the nonsymmetric NBS have been proposed in the context of Rubinstein (1982)'s model, namely considering asymmetry in the parties' discount rates, preferences, or beliefs about determinants in the environment (Binmore et al. 1986; Muthoo 1999). The present work adds to this literature a novel pathway to link bargaining to the nonsymmetric NBS, i.e., the ratio of the confidence factors. Moreover, in contrast to existing models explaining the nonsymmetric NBS, our model considers the entire bargaining process consisting of several rounds, rather than the typically studied situation in which one party offers an equilibrium solution and the other party immediately accepts it.

If the utilities are not concave, we can still show that any local maximum of the nonsymmetric Nash product is a locally stable agreement, but there might be many such maxima and the bargaining process might miss the global maximum (or even a local maximum). The simulation study in Sect. 4 sheds light on the possible outcomes of bargaining processes in such situations. The results show that the process misses the global maximum in a considerable number of cases, especially if there is a large imbalance in confidence between the parties. A local maximum is reached in most of the cases, even when confidence values are highly unbalanced.

The fact that the global optimum is missed, however, might not entail a large loss. The simulation results indicate that in most cases the bargaining outcome reached by the parties is fairly high even in unbalanced cases, both in terms of ratio of the nonsymmetric Nash function (compared to the global maximum) and in terms of joint utility (sum of utilities). The loss in both cases decreases when confidence gets more balanced.

Finally, the theoretical results obtained for the concave utilities case, as well as the simulation results for general utility functions, confirm the general conclusion that the more confident party has an advantage. The party with higher confidence will always get a result which is better than the (symmetric) NBS (and the reverse occurs for the other party).

This article suggests multiple paths for future research, both to address its acknowledged limitations and to exploit its application to empirical research. One such path will possibly be concerned with the confidence parameters γ_s and γ_b. Future theoretical and behavioral studies can address the way these parameters are driven by bargaining experience, expectations about the appearance of attractive outside options, or other elements defining bargaining strength (time pressure, linkage with other negotiations, etc).

One of the limitations of the present work is that we have assumed these parameters are constant throughout the process, i.e., we treat them as an attribute of each negotiator. This is a simplification, because subjective expectations may as well be influenced by what one party is observing in the behavior of the other party. Future developments can thus consider that these parameters change during the bargaining process, extending the model with a function to update confidence. Actually, it is not hard to conjecture what happens for a plausible class of update functions: if we reasonably model that in an imbalanced situation the more confident party becomes even more confident over time, while the reverse occurs for the other party, then the process and the outcome become even more imbalanced than our model stipulates. Thus, if confidence is affected by the concession pattern, our results can be seen as a lower bound of the outcome for the more confident party and an upper bound for the less confident party.

Let us also note that the assumption of a constant confidence parameter is important mainly in the simulations part of the paper. For the analysis in Sect. 3, as long as confidence does not change while a pair of options is on the table, the results hold for the prediction of who would concede, and by how much, at a given iteration step. If confidence changes afterwards, then it will influence concession behavior in the ensuing round. At end of the process, the results about the outcome hold for the confidence at the time the outcome is reached.

The limitation of not considering outside options can also be addressed in future theoretical and behavioral work to study the interplay between the confidence in reaching a good agreement with the negotiator one is facing and the expectation that an even better agreement might be reached with an outside party that might appear in the meantime. In the negotiation round in which such an outside option appears, it can directly be compared to the expectations and offers on the table. The main modification to the model required by this extension therefore would refer to how expectations about future outcomes are formed.

A third limitation of the model is that it assumes the two parties will have the same subjective estimates of p_d. Extending the model to cover a situation in which the parties have different estimates requires the introduction of additional parameters. Some consequences of such extensions are discussed in section 2.

Additional extensions of the model can explicitly model how negotiators estimate the breakdown risk. This would allow to consider the effect of biases in dealing with probabilities on the negotiation process, as well as of differences between parties in their ability to make that estimate.

Regarding other analytical studies, this type of model can be of interest in the field of negotiation performed by artificial intelligence agents (Jennings et al. 2001). Also, more analytical studies could show that some specific types of non-concave utility functions also lead to a single-peaked nonsymmetric Nash product, as it was possible to observe in the standard Z-H model (Dias and Vetschera 2019a).

The model presented here could also be useful for empirical studies. It provides a connection between negotiator characteristics and bargaining outcomes. Thus it could on the one hand be used to infer negotiator confidence from bargaining outcomes. On the other hand, if measures of confidence (or overconfidence, which has been shown to affect negotiations, see Neale and Bazerman 1985) are available, the model makes clear predictions how these will influence the bargaining process and its outcomes, and these predictions could also be tested empirically.

Appendix

Proofs that b/s space can be partitioned as shown in Fig. 2. This means that for $r < 1$, we have to show the following properties:

- The curve separating regions A and B, i.e. the curve along which $\partial f/\partial s = 0$, is monotonically decreasing.
- The curve separating regions B and C, i.e. the curve along which $f = 0$, passes through the point $(b = 0, s = 1)$ and is monotonically decreasing.
- The curve separating regions C and D, i.e. the curve along which $\partial f/\partial b = 0$, is monotonically increasing.
- All three curves intersect with the line $s = b$ at the same point.

Preliminaries

Lemma 1 *For $b = 0$, $\partial f/\partial s = 0$ when s corresponds to the NBS.*

$$\frac{\partial f}{\partial s} = r[u_s'(s)u_b(s) + u_s(s)u_b'(s)] + (1-r)u_s(b)u_b'(s) \tag{30}$$

Proof
Since $u_s(0) = 0$, (30) reduces to r times $u_s'(s)u_b(s) + u_s(s)u_b'(s)$, which is null if and only if $[u_s(s)u_b(s)]'$ is null. Thus, this is the point which maximizes $u_s(s)u_b(s)$, i.e. the NBS. □

Lemma 2 *For $s = 1$, $\partial f/\partial b = 0$ when b corresponds to the NBS.*

$$\frac{\partial f}{\partial b} = (1-r)u_s'(b)u_b(s) - u_b'(b)u_s(b) - u_b(b)u_s'(b) \tag{31}$$

Proof
Since $u_b(1) = 0$, (31) reduces again to the first derivative of the Nash objective function with respect to b. □

To show the relevant properties of the lines separating the different regions in Fig. 2, we have to consider the second derivatives of f:

Lemma 3 *For $s > b$: $\partial^2 f/\partial^2 b > 0$*

Proof From (31), we obtain

$$\partial^2 f/\partial^2 b = u_s''(b)[(1-r)u_b(s) - u_b(b)] - u_b''(b)u_s(b) - 2u_b'(b)u_s'(b) \tag{32}$$

Since we assume concave utilities, all second derivatives are negative. By definition of the utilities, $u_b'(x) < 0$ and $u_s'(x) > 0$. Therefore the second and third term are negative, since they have also a negative sign, their contribution to the sum is positive. Since $s > b$ and the buyer's utility decreases, the second factor of the first term is negative, so the entire term is positive. □

Lemma 4 *For $s > b$: $\partial^2 f/\partial^2 s < 0$*

Proof From (30), we obtain

$$\partial^2 f/\partial^2 s = u_b''(s)[ru_s(s) + (1-r)u_s(b)] + ru_s''(s)u_b(s) + 2ru_s(s)u_b'(s) \tag{33}$$

By a similar argument as above, the last two terms are negative. The second factor of the first term is a weighted combination of utilities and therefore positive, the first factor is negative, so the entire term is negative. □

Lemma 5 *The sign of $\partial^2 f/\partial b\partial s = \partial^2 f/\partial s\partial b$ depends on r. For $r > 1$, it is positive, for $r < 1$, it is negative.*

Proof From the first derivatives, we obtain

$$\partial^2 f/\partial b\partial s = \partial^2 f/\partial s\partial b = (1-r)u_s'(b)u_b'(s) \tag{34}$$

The product of the two derivatives is negative, thus the sign of $(1-r)$ determines the sign as indicated. □

Separation between regions *A* and *B*

Proposition 5 *The curve in b/s space at which $\partial f/\partial s = 0$ is monotonically decreasing for $r < 1$ and monotonically increasing for $r > 1$*

Proof We have shown in Lemma 4 that $\partial^2 f/\partial^2 s < 0$. For $r < 1$, according to Lemma 5, $\partial^2 f/\partial b\partial s < 0$ Thus, an increase in s must be matched by a decrease in b to keep the value of $\partial f/\partial s$ at zero. For $r > 1$, according to Lemma 5, $\partial^2 f/\partial b\partial s > 0$ Thus, an increase in s must be matched by an increase in b to keep the value at zero. □

Corollary 1 *The point at which this curve intersects the line $s = b$ is above the NBS $(b = s = N)$ for $r > 1$ and below the NBS for $r < 1$.*

Proof As shown in Lemma 2 the curve intersects the boundary $b = 0$ at $s = N$. When b increases, s will increase or decrease according to the above proposition, leading to the indicated outcome. □

Separation between regions *C* and *D*

Proposition 6 *The curve in b/s space at which $\partial f/\partial b = 0$ is monotonically increasing for $r < 1$ and monotonically decreasing for $r > 1$.*

Proof We have shown in Lemma 3 that $\partial^2 f/\partial^2 b > 0$. For $r < 1$, according to Lemma 5, $\partial^2 f/\partial b\partial s < 0$ Thus, an increase in s must be matched by an increase in b to keep the value of $\partial f/\partial b$ at zero. For $r > 1$, according to Lemma 3, $\partial^2 f/\partial b\partial s > 0$ Thus, an increase in s must be matched by a decrease in b to keep the value at zero. □

Corollary 2 *The point at which this curve intersects the line $s = b$ is above the NBS $(b = s = N)$ for $r > 1$ and below the NBS for $r < 1$.*

Proof The curve intersects with $s = 1$ at $b = N$. Decreasing s will lead to an increase or decrease in b according to the above proposition, leading to the indicated outcome. □

Separation between regions *B* and *C*

Proposition 7 *The curve $f = 0$ in b/s space is monotonically decreasing.*

Proof Obviously, $f(0, 1) = 0$, therefore the curve ends at $b = 0, s = 1$. There, both derivatives are negative, any decrease in s must therefore be matched by an increase in b. As we have just shown, the two curves where the two derivatives are zero can only intersect at the line $s = b$. Thus, for $s > b$, there is always a corridor between the two curves in which both derivatives are negative. □

Intersection of separating lines

Proposition 8 *The curves $\partial f/\partial b = 0$ and $\partial f/\partial s = 0$ intersect at a point where $s = b$ holds.*

Proof We can rewrite the two curves from (31) and (30) as

$$\partial f/\partial b = (1 - r)u_s'(b)u_b(s) - N'(b) = 0 \tag{35}$$

and

$$\partial f/\partial s = rN'(s) + (1 - r)u_s(b)u_b'(s) = 0 \tag{36}$$

where $N(x)$ is the Nash objective function $u_s(x)u_b(x)$, and $N'(x)) = u_s'(x)u_b(x) + u_s(x)u_b'(x)$. Now assume that $\partial f/\partial b = 0$ intersects the line $s = b$ at a point where $s = b = x$. Then, we can write (35) as

$$\begin{aligned}
\partial f/\partial b &= (1 - r)u_s'(x)u_b(x) - N'(x) = 0 \\
&\Leftrightarrow (1 - r)u_s'(x)u_b(x) - N'(x) + (1 - r)N'(x) - (1 - r)N'(x) = 0 \\
&\Leftrightarrow (1 - r)u_s'(x)u_b(x) - rN'(x) - (1 - r)N'(x) = 0 \\
&\Leftrightarrow -rN'(x) - (1 - r)u_s(x)u_b'(x) = 0 \\
&\Leftrightarrow \partial f/\partial s = 0.
\end{aligned}$$

Therefore, at $s = b = x$ both $\partial f/\partial b = 0$ and $\partial f/\partial s = 0$. Since we have already shown that both curves are monotonic in the opposite direction, they can intersect only at one point, so there can be no intersection at which $s \neq b$. □

Proposition 9 *The curve where $f = 0$ and the two curves at which the derivatives are zero all intersect with the line $s = b$ at the same point.*

Proof We have already shown that the curves at which the derivatives are zero intersect at the line $s = b$. Note that at the line $s = b$, also $f = 0$ holds. Thus we have an intersection of two lines at which $f = 0$. The line $s = b$ is monotonically increasing, which implies that the derivatives of f with respect to the two variables must have opposite signs. The other curve at which $f = 0$ holds is monotonically decreasing, so along that curve the two derivatives must have the same sign. This can only happen at the same time if both derivatives are zero, so the two other lines at which the derivatives are zero must also go through that point. □

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