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Localic real functions: A general setting ☆

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ABSTRACT

In pointfree topology the lattice-ordered ring of all continuous real functions on a frame L has *not* been a part of the lattice of all lower (or upper) semicontinuous real functions on L just because all those continuities involve different domains. This paper demonstrates a framework in which all those continuous and semicontinuous functions arise (up to isomorphism) as members of the lattice-ordered ring of *all* frame homomorphisms from the frame $\mathfrak{L}(\mathbb{R})$ of reals into $\mathcal{S}(L)$, the dual of the co-frame of all sublocales of L. The lattice-ordered ring $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$ is a pointfree counterpart of the ring $\mathbb{R}^X$ with X a topological space. We thus have a pointfree analogue of the concept of an *arbitrary not necessarily (semi) continuous* real function on L. One feature of this remarkable conception is that one eventually has: *lower semicontinuous* $+$ *upper semicontinuous* $=$ *continuous*. We document its importance by showing how nicely can the insertion, extension and regularization theorems, proved earlier by these authors, be recast in the new setting.

1. Introduction

A [*semi-*] *continuous real function* on a frame L has up to now been understood as a frame homomorphism from the frame $\mathfrak{L}(\mathbb{R})$ of reals into L [as a frame homomorphism (modulo some conditions) from certain subframes of $\mathfrak{L}(\mathbb{R})$ into L] (definitions are given below). The main disadvantage of these continuities is that the involved functions have different domains. Also, the parallel between functions and sets in point-set topology does not yet have a fine counterpart in pointfree topology which suffers of not having the concept of an *arbitrary not necessarily continuous* frame real function. After this paper, the following quotation from Gillman and Jerison [7, Chapter 1] will make sense in the pointfree setting:

> The set $\mathrm{C}(X)$ of all continuous, real-valued functions on a topological space X will be provided with an algebraic structure and an order structure. Since their definitions do not involve continuity, we begin by imposing these structures on the collection $\mathbb{R}^X$ of all functions from X into the set $\mathbb{R}$ of real numbers. [...] In fact, it is clear that $\mathbb{R}^X$ is a commutative ring with unity element (provided that X is non empty). [...] The partial ordering on $\mathbb{R}^X$ is defined by: $f \geq g$ if and only if $f(x) \geq g(x)$ for all $x \in X$. [...] The set of all continuous functions from the topological space X into the space $\mathbb{R}$ is denoted $\mathrm{C}(X)$. [...] Therefore $\mathrm{C}(X)$ is a commutative ring, a subring of $\mathbb{R}^X$.

The localic analogue of $\mathbb{R}^X$ will be just the lattice-ordered ring $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$ where $\mathcal{S}(L)$ is the dual of the co-frame of all sublocales of L. Members of this ring can be though as *arbitrary not necessarily continuous* real functions on L. This is

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reminiscent of dealing with (not necessarily continuous) real functions $X \to \mathbb{R}$ as with continuous functions $\mathcal{D}(X) \to \mathbb{R}$ where $\mathcal{D}(X)$ is the discrete space on the underlying set of X. But we alert the reader not to think that this is just putting $M = \mathcal{S}(L)$ in $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), M)$ (which is a lattice ordered ring for any frame M [2]). It is deeper: having $\mathfrak{L}(\mathbb{R})$ as a common domain and $\mathcal{S}(L)$ as a common codomain, the structure of $\mathcal{S}(L)$ is rich enough to allow to distinguish all the previously mentioned types of continuities. The relations of minorization and majorization, which have previously been used to relate the incomparable lower and upper semicontinuous functions, are no longer needed, for we just have an ordering in $\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$. Also, in contrast to [9], all localic real functions will now have their lower and upper regularizations. Finally, we show how nicely can the insertion and extension theorems, proved in [8–10,17], be recast in our new setting.

Convention. If not otherwise stated, L stands for an arbitrary frame.

2. The frame of sublocales

Our references for frames are [14,19]. Regarding sublocales we follow [18]. We recall that the category Frm of *frames* has as objects those complete lattices L in which $a \wedge \bigvee B = \bigvee\{a \wedge b : b \in B\}$ for all $a \in L$ and $B \subseteq L$. Universal bounds are denoted by 0 and 1. Morphisms, called *frame homomorphisms*, are maps between frames which preserve arbitrary joins and finite meets. The set of all morphisms from L into M is denoted by $\mathsf{Frm}(L, M)$. The category of *locales* is the opposite category of Frm.

Being a Heyting algebra, each frame L has the implication $\to$ satisfying: $a \wedge b \le c$ iff $a \le b \to c$. The *pseudocomplement* of an $a \in L$ is $a^* = a \to 0 = \bigvee\{b \in L : a \wedge b = 0\}$. Then: $a \le a^{**}$ and $(\bigvee A)^* = \bigwedge_{a\in A} a^*$ for all $A \subseteq L$. In particular, $(\cdot)^*$ is order-reversing.

An $S \subseteq L$ is a *sublocale* of L if, whenever $A \subseteq S$, $a \in L$ and $b \in S$, then $\bigwedge A \in S$ and $a \to b \in S$. The set of all sublocales of L forms a co-frame under inclusion, in which arbitrary meets coincide with intersection, $\{1\}$ is the bottom, and L is the top.

Convention. For notational reasons, we make the co-frame of all sublocales into a frame $\mathcal{S}(L)$ by considering the dual ordering: $S_1 \le S_2$ iff $S_2 \subseteq S_1$.

Thus, given $\{S_i \in \mathcal{S}(L) : i \in I\}$, we have $\bigvee_{i\in I} S_i = \bigcap_{i\in I} S_i$ and $\bigwedge_{i\in I} S_i = \{\bigwedge A : A \subseteq \bigcup_{i\in I} S_i\}$. Also, $\{1\}$ is the top and L is the bottom in $\mathcal{S}(L)$ that we simply denote by 1 and 0, respectively. The pseudocomplement of S in $\mathcal{S}(L)$ will as standard be denoted by S^*. For any $a \in L$, the sets

$$\mathfrak{c}(a) = \ \uparrow a \quad \text{and} \quad \mathfrak{o}(a) = \{a \to b : b \in L\}$$

are sublocales of L called, respectively *open* and *closed*. We shall freely use the following properties:

Properties 2.1. *For all $a, b \in L$ and $A \subseteq L$:*

(1) $\mathfrak{c}(a) \le \mathfrak{c}(b)$ *if and only if* $a \le b$,
(2) $\mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \wedge \mathfrak{c}(b)$,
(3) $\mathfrak{c}(\bigvee A) = \bigvee_{a\in A} \mathfrak{c}(a)$,
(4) $\mathfrak{c}(\bigwedge A) \le \bigwedge_{a\in A} \mathfrak{c}(a)$.
(5) $\mathfrak{o}(a) \ge \mathfrak{c}(b)$ *if and only if* $a \wedge b = 0$,
(6) $\mathfrak{o}(a) \le \mathfrak{c}(b)$ *if and only if* $a \vee b = 1$,
(7) $\mathfrak{c}(a) = \mathfrak{o}(b)$ *if and only if* a *and* b *are complements of each other,*
(8) $\mathfrak{c}(a) \vee \mathfrak{o}(a) = 1$ *and* $\mathfrak{c}(a) \wedge \mathfrak{o}(a) = 0$.

Thus $\mathfrak{c}(a)$ and $\mathfrak{o}(a)$ are complements of each other in $\mathcal{S}(L)$. Note also that the map $a \mapsto \mathfrak{c}(a)$ is a frame embedding $L \hookrightarrow \mathcal{S}(L)$. The subframe of $\mathcal{S}(L)$ consisting of all closed sublocales will be denoted by $\mathfrak{c}L$. Clearly, L and $\mathfrak{c}L$ are isomorphic.

Given a sublocale S of L, its *closure* and *interior* are defined, respectively, by

$$\overline{S} = \bigvee\{\mathfrak{c}(a) : \mathfrak{c}(a) \le S\} = \mathfrak{c}\left(\bigwedge S\right)$$

and

$$S^\circ = \bigwedge\{\mathfrak{o}(a) : S \le \mathfrak{o}(a)\}.$$

The following is well-known:

Proposition 2.2. *Let $S, T \in \mathcal{S}(L)$, $a \in L$ and $A \subseteq L$. Then:*

(1) $\overline{1} = 1$, $\overline{S} \le S$, $\overline{\overline{S}} = \overline{S}$, *and* $\overline{S \wedge T} = \overline{S} \wedge \overline{T}$,
(2) $0^\circ = 0$, $S^\circ \ge S$, $S^{\circ\circ} = S^\circ$, *and* $(S \vee T)^\circ = S^\circ \vee T^\circ$,
(3) $S^\circ = (\overline{S^*})^* = \mathfrak{o}(\bigwedge S^*)$,
(4) $\mathfrak{c}(a)^\circ = \mathfrak{o}(a^*)$,
(5) $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*)$.

3. Frames of reals and their continuity notions

Notation. We write $L = \langle A \rangle$ if L is generated by $A \subseteq L$.

There are various equivalent definitions of the frame of reals (see e.g. [14,2,3]). In [2,3], the *frame* $\mathfrak{L}(\mathbb{R})$ *of reals* is the frame generated by all pairs $(p, q) \in \mathbb{Q} \times \mathbb{Q}$ satisfying the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \le r < q \le s$,
(R3) $(p, q) = \bigvee\{(r, s) : p < r < s < q\}$,
(R4) $\bigvee_{p,q\in\mathbb{Q}}(p, q) = 1$.

One writes: $(p, -) = \bigvee_{q\in\mathbb{Q}}(p, q)$ and $(-, q) = \bigvee_{p\in\mathbb{Q}}(p, q)$.

As we shall also deal with frames of lower and upper reals, we take $(r, -)$ and $(-, r)$ as primitive notions. We thus adopt the equivalent description of $\mathfrak{L}(\mathbb{R})$ proposed in [16]. Specifically, the *frame of reals* $\mathfrak{L}(\mathbb{R})$ is the one having generators of the form $(r, -)$ and $(-, r)$ subject to the following relations:

(r1) $(r, -) \wedge (-, s) = 0$ whenever $r \ge s$,
(r2) $(r, -) \vee (-, s) = 1$ whenever $r < s$,
(r3) $(r, -) = \bigvee_{s>r}(s, -)$,
(r4) $(-, r) = \bigvee_{s<r}(-, s)$,
(r5) $\bigvee_{r\in\mathbb{Q}}(r, -) = 1$,
(r6) $\bigvee_{r\in\mathbb{Q}}(-, r) = 1$.

With $(p, q) = (p, -) \wedge (-, q)$ one goes back to (R1)–(R4). So, besides $\mathfrak{L}(\mathbb{R})$, we have its subframes of *upper* and *lower reals*:

$$\mathfrak{L}_u(\mathbb{R}) = \langle\{(r, -) : r \in \mathbb{Q}, (r, -) \text{ satisfy (r3) and (r5) for all } r \in \mathbb{Q}\}\rangle,$$
$$\mathfrak{L}_l(\mathbb{R}) = \langle\{(-, r) : r \in \mathbb{Q}, (-, r) \text{ satisfy (r4) and (r6) for all } r \in \mathbb{Q}\}\rangle.$$

When dropping (r5) and (r6), we get the *extended* variants of frames just introduced, namely: $\mathfrak{L}(\overline{\mathbb{R}})$, $\mathfrak{L}_u(\overline{\mathbb{R}})$, and $\mathfrak{L}_l(\overline{\mathbb{R}})$. Members of

$$\overline{\mathrm{lsc}}(L) = \mathsf{Frm}(\mathfrak{L}_u(\overline{\mathbb{R}}), L),$$
$$\overline{\mathrm{usc}}(L) = \mathsf{Frm}(\mathfrak{L}_l(\overline{\mathbb{R}}), L),$$
$$\overline{\mathrm{c}}(L) = \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), L)$$

are called *extended*, respectively, *lower semicontinuous, upper semicontinuous*, and *continuous real functions* on L, while members of

$$\mathrm{lsc}(L) = \left\{ f \in \mathsf{Frm}(\mathfrak{L}_u(\mathbb{R}), L) : \bigvee_{r\in\mathbb{Q}} \mathfrak{o}(f(r, -)) = 1 \right\},$$
$$\mathrm{usc}(L) = \left\{ f \in \mathsf{Frm}(\mathfrak{L}_l(\mathbb{R}), L) : \bigvee_{r\in\mathbb{Q}} \mathfrak{o}(f(-, r)) = 1 \right\},$$
$$\mathrm{c}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$$

are called *lower semicontinuous, upper semicontinuous*, and *continuous real functions* on L [13].

Remark. The extra conditions in the definitions of $\mathrm{lsc}(L)$ and $\mathrm{usc}(L)$ come from [13], to which we refer for their role. The latter has been then exhibited in [8–10]. Their role will also be seen in this paper (cf. the proof of (4) of Proposition 6.1). In contrast to [13], we recall that, in this paper, we make the co-frame of sublocales into the frame $\mathcal{S}(L)$.

Partial orderings. (1) The set $\overline{\mathrm{lsc}}(L)$ is partially ordered by

$$f_1 \le f_2 \quad \Leftrightarrow \quad f_1(r, -) \le f_2(r, -) \quad \text{for all } r \in \mathbb{Q}.$$

Under this ordering, $\overline{\mathrm{lsc}}(L)$ is closed under finite meets and arbitrary nonempty joins: $(f_1 \wedge f_2)(r, -) = f_1(r, -) \wedge f_2(r, -)$ and $(\bigvee \mathcal{F})(r, -) = \bigvee_{f\in\mathcal{F}} f(r, -)$, where $\varnothing \ne \mathcal{F} \subseteq \overline{\mathrm{lsc}}(L)$. The constant map with value 1 is the top, while there is no bottom in $\overline{\mathrm{lsc}}(L)$.

(2) The set $\overline{\mathrm{usc}}(L)$ is partially ordered by the reverse pointwise ordering:

$$f_1 \le f_2 \quad \Leftrightarrow \quad f_2(-, r) \le f_1(-, r) \quad \text{for all } r \in \mathbb{Q},$$

under which it is closed with respect to finite joins and arbitrary nonempty meets: $(f_1 \vee f_2)(-, q) = f_1(-, q) \wedge f_2(-, q)$ and $(\bigwedge \mathcal{F})(-, r) = \bigvee_{f\in\mathcal{F}} f(-, r)$, where $\varnothing \ne \mathcal{F} \subseteq \overline{\mathrm{usc}}(L)$. The constant map with value 1 is the bottom element, while there is no top element in $\overline{\mathrm{usc}}(L)$.

(3) The set $\overline{\mathrm{c}}(L)$ is partially ordered by

$$f_1 \le f_2 \quad \Leftrightarrow \quad f_{1|\mathfrak{L}_u(\overline{\mathbb{R}})} \le f_{2|\mathfrak{L}_u(\overline{\mathbb{R}})} \quad \Leftrightarrow \quad f_{2|\mathfrak{L}_l(\overline{\mathbb{R}})} \le f_{1|\mathfrak{L}_l(\overline{\mathbb{R}})}.$$

Remark. There is an order-isomorphism $\text{-}(\cdot) : \overline{\text{lsc}}(L) \to \overline{\text{usc}}(L)$ defined by $(\text{-}f)(-, r) = f(\text{-}r, -)$ for all $r \in \mathbb{Q}$. When restricted to $\text{lsc}(L)$ it becomes an isomorphism from $\text{lsc}(L)$ onto $\text{usc}(L)$. Its inverse, denoted by the same symbol, maps a $g \in \overline{\text{usc}}(L)$ into $\text{-}g \in \overline{\text{lsc}}(L)$ defined by $(\text{-}g)(r, -) = g(-, \text{-}r)$ for all $r \in \mathbb{Q}$, etc.

4. Generating frame homomorphisms by scales

A way of generating continuous real functions on frames by the so-called scales has been described in detail in [2] with $\mathfrak{L}(\mathbb{R})$ being generated by pairs of rationals satisfying the relations (R1)–(R4) (cf. also [14, p. 127]). In what follows we decompose the investigations of [2] into two pieces so as to have ways of generating all the types of real functions on frames by means of scales. In what follows p, q, r, s stand for rationals.

Definition 4.1. A family $\mathcal{C} = \{c_r : r \in \mathbb{Q}\} \subseteq L$ is called an *extended scale* in L if $c_r \vee c_s^* = 1$ whenever $r < s$. An extended scale is called a *scale* if $\bigvee_{r\in\mathbb{Q}} c_r = 1 = \bigvee_{r\in\mathbb{Q}} c_r^*$.

Remark 4.2. An extended scale $\mathcal{C}$ is necessarily antitone. However, if $\mathcal{C}$ consists of complemented elements, then $\mathcal{C}$ is an extended scale if and only if it is antitone. Indeed, in such a case one has $c_r \vee c_s^* \geq c_r \vee c_r^* = 1$ whenever $r < s$.

Lemma 4.3. *Let $\mathcal{C} = \{c_r : r \in \mathbb{Q}\}$ be an extended scale in L and let*

$$f(r, -) = \bigvee_{s>r} c_s \quad \text{and} \quad f(-, r) = \bigvee_{s<r} c_s^*$$

for all $r \in \mathbb{Q}$. Then the following hold:

(1) *The above two formulas determine an $f \in \overline{\text{c}}(L)$;*
(2) *If $\mathcal{C}$ is a scale, then $f \in \text{c}(L)$.*

Proof. (1) We must check that (r1)–(r4) hold for the extended scale $\mathcal{C}$. To show (r1), let $r \geq s$. Then $f(r, -) \wedge f(-, s) \leq c_r \wedge c_s^* \leq c_r \wedge c_r^* = 0$. As for (r2), if $r < p < q < s$, then $f(r, -) \vee f(-, s) \geq c_p \vee c_q^* = 1$. To show (r3), we have $\bigvee_{s>r} f(s, -) = \bigvee_{s>r} \bigvee_{p>s} c_p = \bigvee_{p>r} c_p = f(r, -)$. Analogously, one checks (r4).

(2) Let $\mathcal{C}$ be a scale. To show (r5), we have $\bigvee_r f(r, -) = \bigvee_r \bigvee_{s>r} c_s = 1$, and similarly for (r6). □

Lemma 4.4. *Let $f, g \in \overline{\text{c}}(L)$ be generated by the extended scales $\{c_r : r \in \mathbb{Q}\}$ and $\{d_r : r \in \mathbb{Q}\}$, respectively. Then:*

(1) *$f(r, -) \leq c_r \leq f(-, r)^*$ for all $r \in \mathbb{Q}$;*
(2) *$f \leq g$ if and only if $c_r \leq d_s$ whenever $r > s$ in $\mathbb{Q}$.*

Proof. (1) We have $f(r, -) \leq c_r \leq c_r^{**} \leq \bigwedge_{s<r} c_s^{**} = (\bigvee_{s<r} c_s^*)^* = f(-, r)^*$.

(2) If $r > s$, then $f(-, r)^* = f(-, r)^* \wedge (f(s, -) \vee f(-, r)) = f(-, r)^* \wedge f(s, -)$. So, if $f \leq g$, then $c_r \leq f(-, r)^* \leq f(s, -) \leq g(s, -) \leq d_s$. For the converse, let $q > r > s$. Since $d_s^* \leq c_r^*$, we have $d_s^* \leq \bigvee_{r<q} c_r^* = f(-, q)$. So, $g(-, q) = \bigvee_{s<q} d_r^* \leq f(-, q)$, i.e. $f \leq g$. □

Lemma 4.5. *Let $\{d_r : r \in \mathbb{Q}\} \subseteq L$ be antitone. Then:*

(1) *$\{\mathfrak{c}(d_r) : r \in \mathbb{Q}\}$ is an extended scale in $\mathcal{S}(L)$;*
(2) *If $\bigvee_{r\in\mathbb{Q}} \mathfrak{c}(d_r) = 1 = \bigvee_{r\in\mathbb{Q}} \mathfrak{c}(d_r)^*$, then $\{\mathfrak{c}(d_r) : r \in \mathbb{Q}\}$ is a scale in $\mathcal{S}(L)$.*

Proof. (1) follows immediately by Remark 4.2. One detail for (2) is that $\bigvee_r \mathfrak{c}(d_r) = \mathfrak{c}(\bigvee_r d_r) = \mathfrak{c}(1) = 1$. □

5. Localic real functions

In general topology one sometimes sees the phrase: *Let X be a topological space and let f be an arbitrary not necessarily continuous real-valued function on X*. In this section that will become possible in the pointfree setting.

Notation. We let

$$\text{F}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L)) = \text{c}(\mathcal{S}(L)),$$
$$\overline{\text{F}}(L) = \mathsf{Frm}(\mathfrak{L}(\overline{\mathbb{R}}), \mathcal{S}(L)) = \overline{\text{c}}(\mathcal{S}(L)).$$

Definition 5.1. An $F \in \text{F}(L)$ will be called an *arbitrary real function* on L. We shall say that F is:

(1) *lower semicontinuous* if $F(r, -)$ is a closed sublocale for all r, i.e. $F(\mathfrak{L}_u(\mathbb{R})) \subseteq \mathfrak{c}L$;
(2) *upper semicontinuous* if $F(-, r)$ is a closed sublocale for all r, i.e. $F(\mathfrak{L}_l(\mathbb{R})) \subseteq \mathfrak{c}L$;
(3) *continuous* if $F(p, q)$ is a closed sublocale for all p, q. i.e. $F(\mathfrak{L}(\mathbb{R})) \subseteq \mathfrak{c}L$.

Notation. We denote by

$$\mathrm{LSC}(L),\ \mathrm{USC}(L) \text{ and } \mathrm{C}(L)$$

the collections of all lower semicontinuous, upper semicontinuous, and continuous members of $F(L)$. If we replace $\mathfrak{L}_u(\mathbb{R})$, $\mathfrak{L}_l(\mathbb{R})$, and $\mathfrak{L}(\mathbb{R})$ by $\mathfrak{L}_u(\overline{\mathbb{R}})$, $\mathfrak{L}_l(\overline{\mathbb{R}})$, and $\mathfrak{L}(\overline{\mathbb{R}})$ in, respectively, (1), (2), and (3) above, we get the collections

$$\overline{\mathrm{LSC}}(L),\ \overline{\mathrm{USC}}(L),\ \text{ and } \overline{\mathrm{C}}(L)$$

of all *extended* lower semicontinuous, upper semicontinuous, and continuous members of $\overline{\mathrm{F}}(L)$. Of course, one has

$$\mathrm{C}(L) = \mathrm{LSC}(L) \cap \mathrm{USC}(L) \quad \text{and} \quad \overline{\mathrm{C}}(L) = \overline{\mathrm{LSC}}(L) \cap \overline{\mathrm{USC}}(L).$$

All the above collections of morphisms are partially ordered according to the definition of partial orderings in $\overline{\mathrm{lsc}}(L)$, $\overline{\mathrm{usc}}(L)$, and $\overline{\mathrm{c}}(L)$ where L is replaced by $\mathcal{S}(L)$. Thus, given $F, G \in \overline{\mathrm{F}}(L)$, one has

$$\begin{aligned} F \le G &\Leftrightarrow F(r,-) \le G(r,-) \quad \text{for all } r \in \mathbb{Q} \\ &\Leftrightarrow G(-,r) \le F(-,r) \quad \text{for all } r \in \mathbb{Q}. \end{aligned}$$

Given a complemented sublocale S of L, we define the *characteristic map* $\chi_S : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ by

$$\chi_S(r,-) = \begin{cases} 1 & \text{if } r < 0 \\ S^* & \text{if } 0 \le r < 1 \\ 0 & \text{if } r \ge 1 \end{cases} \quad \text{and} \quad \chi_S(-,r) = \begin{cases} 0 & \text{if } r \le 0 \\ S & \text{if } 0 < r \le 1 \\ 1 & \text{if } r > 1. \end{cases}$$

for each $r \in \mathbb{Q}$. We then have: $\chi_S \in \mathrm{LSC}(L)$ (resp. $\mathrm{USC}(L)$) iff S is open (resp. closed). Consequently, $\chi_S \in \mathrm{C}(L)$ iff S is clopen.

6. Embedding of lsc(L) and usc(L) into F(L)

In this section, we consider order-embeddings of $\overline{\mathrm{lsc}}(L)$, $\overline{\mathrm{usc}}(L)$, and $\overline{\mathrm{c}}(L)$ into $\overline{\mathrm{F}}(L)$. Let $f \in \overline{\mathrm{lsc}}(L)$. Then $\{f(r,-) : r \in \mathbb{Q}\}$ is antitone and, thus, $\{\mathfrak{c}(f(r,-))\}$ is an extended scale in $\mathcal{S}(L)$ (cf. Remark 4.2). Using Lemma 4.3, we define

$$\Sigma : \overline{\mathrm{lsc}}(L) \to \overline{\mathrm{LSC}}(L)$$

by the following two formulas:

$$\Sigma(f)(r,-) = \bigvee_{s>r} \mathfrak{c}(f(s,-)) \quad \text{and} \quad \Sigma(f)(-,r) = \bigvee_{s<r} \mathfrak{c}(f(s,-))^*.$$

As $\Sigma(f)(r,-) = \mathfrak{c}(f(r,-)) \in \mathfrak{c}L$, we indeed have $\Sigma(f) \in \overline{\mathrm{LSC}}(L)$. Dually, we define

$$\Upsilon : \overline{\mathrm{usc}}(L) \to \overline{\mathrm{USC}}(L)$$

by

$$\Upsilon(f) = -\Sigma(-f).$$

An easy calculation shows that

$$\Upsilon(f)(r,-) = \bigvee_{s>r} \mathfrak{c}(f(-,s))^* \quad \text{and} \quad \Upsilon(f)(-,r) = \bigvee_{s<r} \mathfrak{c}(f(-,s)).$$

Since $\Upsilon(f)(-,r) = \mathfrak{c}(f(-,r)) \in \mathfrak{c}L$, we indeed have $\Upsilon(f) \in \overline{\mathrm{USC}}(L)$. Observe that $\{\mathfrak{c}(f(-,r))^* : r \in \mathbb{Q}\}$ is an extended scale which generates $\Upsilon(f)$ (cf. Lemma 4.3 and note that $\mathfrak{c}(f(-,r))^{**} = \mathfrak{c}(f(-,r))$).

Finally, using Σ and Υ, we define

$$\Psi : \overline{\mathrm{c}}(L) \to \overline{\mathrm{C}}(L)$$

by

$$\Psi(f)(p,q) = \Sigma(f)(p,-) \wedge \Upsilon(f)(-,q).$$

Proposition 6.1. *The following assertions hold:*

(1) Σ *is a lattice isomorphism preserving arbitrary nonempty joins;*
(2) Υ *is a lattice isomorphism preserving arbitrary nonempty meets;*
(3) Ψ *is a lattice isomorphism;*
(4) *The restrictions* $\Sigma_{|\mathrm{lsc}(L)}$, $\Upsilon_{|\mathrm{lsc}(L)}$, *and* $\Psi_{|\mathrm{c}(L)}$ *take values in* $\mathrm{LSC}(L)$, $\mathrm{USC}(L)$ *and* $\mathrm{C}(L)$, *respectively, and have the corresponding properties.*

Proof. To show (1), we first notice that Σ is clearly injective. For surjectivity: for any $G \in \overline{\mathrm{LSC}}(L)$, we have $\Sigma(g) = G$ with $g(r, -) = \bigwedge G(r, -)$ (i.e. $g = \mathfrak{c}^{-1} \circ G$ where $\mathfrak{c} : L \to \mathfrak{c}L$ is the frame isomorphism sending a to $\mathfrak{c}(a)$). Also, we have $\Sigma(f \wedge g)(r, -) = \Sigma(f)(r, -) \wedge \Sigma(g)(r, -)$ as well as

$$\Sigma\left(\bigvee \mathcal{F}\right)(r, -) = \mathfrak{c}\left(\left(\bigvee \mathcal{F}\right)(r, -)\right) = \bigvee_{f \in \mathcal{F}} \mathfrak{c}(f(r, -)) = \bigvee_{f \in \mathcal{F}} \Sigma(f)(r, -).$$

Assertion (2) follows from (1), while combining (1) and (2) yields (3). Now, we move to the restriction $\Sigma_{|\mathrm{lsc}(L)}$. Assume $f \in \mathrm{lsc}(L)$. Then $\bigvee_{r \in \mathbb{Q}} \Sigma(f)(r, -) = \bigvee_{r \in \mathbb{Q}} \mathfrak{c}(f(r, -)) = \mathfrak{c}(1) = 1$ and

$$\bigvee_{r \in \mathbb{Q}} \Sigma(f)(-, r) = \bigvee_{r \in \mathbb{Q}} \bigvee_{s < r} \mathfrak{c}(f(s, -))^* = \bigvee_{r \in \mathbb{Q}} \mathfrak{o}(f(r, -)) = 1$$

(the latter equality is just the extra condition defining lower semicontinuity). Thus $\Sigma(f) \in \mathrm{LSC}(L)$. The remaining cases follow from what has just been proved. □

Due to the fact that members of $\overline{\mathrm{lsc}}(L)$ and $\overline{\mathrm{usc}}(L)$ have different domains, they have so far been compared in terms of the *minorization* and *majorization relations*. We shall now show that after embedding $\overline{\mathrm{lsc}}(L)$ and $\overline{\mathrm{usc}}(L)$ into $\overline{\mathrm{F}}(L)$ those relations become superfluous. We first recall that if $f \in \overline{\mathrm{lsc}}(L)$ and $g \in \overline{\mathrm{usc}}(L)$, then one says that f *minorizes* g (written: $f \lhd g$) iff

$$f(r, -) \wedge g(-, s) = 0 \quad \text{for all } r > s \text{ in } \mathbb{Q}.$$

Clearly, $f \lhd g$ if and only if $f(r, -) \leq g(-, r)^*$ for all $r \in \mathbb{Q}$. Further, one says that f *majorizes* g (written: $f \blacktriangleright g$) iff

$$f(r, -) \vee g(-, s) = 1 \quad \text{for all } r < s \text{ in } \mathbb{Q}.$$

Proposition 6.2. *Let $f \in \overline{\mathrm{lsc}}(L)$ and $g \in \overline{\mathrm{usc}}(L)$. Then the following hold:*

(1) *$f \lhd g$ if and only if $\Sigma(f) \leq \Upsilon(g)$;*
(2) *$f \blacktriangleright g$ if and only if $\Sigma(f) \geq \Upsilon(g)$.*

Proof. (1) Recall that $\Sigma(f)$ and $\Upsilon(g)$ are generated by the extended scales $\{\mathfrak{c}(f(r, -)\}$ and $\{\mathfrak{c}(g(-, r))^*\}$, respectively. By the definition of $\lhd$ and Properties 2.1(5) we have $f \lhd g$ if and only if $\mathfrak{c}(f(s, -)) \leq \mathfrak{c}(g(-, r))^*$ whenever $r > s$, which on account of Lemma 4.4(2) is equivalent to the statement that $\Sigma(f) \leq \Upsilon(g)$.

(2) A similar argument applies except that the appeal to (5) of Properties 2.1 is replaced by an application (6) of Properties 2.1. □

We close this section by providing relations that hold between the characteristic functions [13] $l_a \in \mathrm{lsc}(L)$ and $u_a \in \mathrm{usc}(L)$, $a \in L$, defined as follows:

$$l_a(r, -) = \begin{cases} 1 & \text{if } r < 0 \\ a & \text{if } 0 \leq r < 1 \\ 0 & \text{if } r \geq 1 \end{cases} \quad \text{and} \quad u_a(-, r) = \begin{cases} 0 & \text{if } r \leq 0 \\ a & \text{if } 0 < r \leq 1 \\ 1 & \text{if } r > 1. \end{cases}$$

Properties 6.3. *For each $a \in L$ we have $\Sigma(l_a) = \chi_{\mathfrak{o}(a)}$ and $\Upsilon(u_a) = \chi_{\mathfrak{c}(a)}$.*

7. Semicontinuous regularizations of localic real functions

We recall from general topology that, given a topological space X and an *arbitrary not necessarily continuous function* $f : X \to \mathbb{R}$ one defines its lower and upper regularizations (also called lower and upper limit functions) as follows: $f_*(x) = \bigvee_{U \in \mathbb{U}_x} \bigwedge f(U)$ and $f^*(x) = \bigwedge_{U \in \mathbb{U}_x} \bigvee f(U)$ for all $x \in X$ where $\mathbb{U}_x$ is the system of all open neighbourhoods of x. Clearly, $f^* = -(-f)_*$ and both f_* and f^* may take values in $\overline{\mathbb{R}}$ (see [1,4,20], as well as [11,12], for the lattice-valued and the domain-valued cases, respectively).

In [9], some effort has been made to have these concepts for frame real functions but with serious limitations. Now, we are in a position to overcome all those obstacles and have a nice theory quite analogous to the standard one. We have chosen F° and F^- to denote the lower and upper regularization of F, rather than the standard notation F_* and F^*, in order to avoid confusion with the well established notation in pointfree topology (cf. [14, p. 40]). In fact, our notation emphasizes the analogy between lower and upper regularizations and interior and closure operators (cf. Propositions 7.3, 7.4 and Properties 7.10). The following is actually a repetition of Lemma 4.5 in the context of antitone subfamilies of $\mathcal{S}(L)$.

Lemma 7.1. *Let $\{S_r : r \in \mathbb{Q}\} \subseteq \mathcal{S}(L)$ be antitone. Then:*

(1) *$\{\overline{S_r} : r \in \mathbb{Q}\}$ is an extended scale in $\mathcal{S}(L)$;*
(2) *If $\bigvee_{r \in \mathbb{Q}} \overline{S_r} = 1$ and $\bigvee_{r \in \mathbb{Q}} (\overline{S_r})^* = 1$, then it is a scale in $\mathcal{S}(L)$.*

In particular, if $F \in \overline{\mathrm{F}}(L)$, then the assignment $r \mapsto F(r,-)$ is antitone and we, thus, have an extended scale $\{\overline{F(r,-)}\}$. Moreover, when $F \in \mathrm{F}(L)$, then

$$\bigvee_{r\in\mathbb{Q}} (\overline{F(r,-)})^* \geq \bigvee_{r\in\mathbb{Q}} F(r,-)^* \geq \bigvee_{r\in\mathbb{Q}} F(-,r) = 1.$$

To motivate our concepts of lower and upper regularizations of an arbitrary localic real function, we recall from [11, Proposition 5.3] or [12, Proposition 4.8], that if $f : X \to \mathbb{R}$ is an arbitrary function (where X is a topological space) which is generated by an antitone family $\{F_r : r \in \mathbb{Q}\}$, that is: $f(x) = \sup\{r \in \mathbb{Q} : x \in F_r\}$, then the lower (resp., upper) regularization f_* (resp., f^*) of f is generated by the family $\{\overline{F_r} : r \in \mathbb{Q}\}$ (resp., $\{{F_r}^\circ : r \in \mathbb{Q}\}$). We now state the following:

Definition 7.2. The *lower regularization* F° of $F \in \overline{\mathrm{F}}(L)$ is defined by:

$$F^\circ(r,-) = \bigvee_{s>r} \overline{F(s,-)} \quad \text{and} \quad F^\circ(-,r) = \bigvee_{s<r} (\overline{F(s,-)})^*.$$

Dually, the *upper regularization* F^- of F is defined by

$$F^- = \text{-}(\text{-}F)^\circ.$$

An easy calculation gives:

$$F^-(r,-) = \bigvee_{s>r} (\overline{F(-,s)})^* \quad \text{and} \quad F^-(-,r) = \bigvee_{s<r} \overline{F(-,s)}.$$

The following proposition shows that $(\cdot)^\circ : \overline{\mathrm{F}}(L) \to \overline{\mathrm{LSC}}(L)$ is actually an interior-like operator.

Proposition 7.3. *The following hold for all $F, G \in \overline{\mathrm{F}}(L)$:*

(1) $\top^\circ = \top$, *where* $\top(-,r) = 0$ *for all* $r \in \mathbb{Q}$,
(2) $F^\circ \leq F$,
(3) $F^{\circ\circ} = F^\circ$,
(4) $(F \wedge G)^\circ = F^\circ \wedge G^\circ$.

Proof. (1) If $s < r$, $\top(s,-) \vee \top(-,r) = 1$, so that $\top(s,-) = 1$ for all s. Thus, $\top^\circ(-,r) = \bigvee_{s<r} (\overline{\top(s,-)})^* = (\overline{1})^* = 0 = \top(-,r)$ for all r.

(2) We have $F^\circ(r,-) = \bigvee_{s>r} \overline{F(s,-)} \leq \bigvee_{s>r} F(s,-) = F(r,-)$, hence $F^\circ \leq F$.

(3) We only need to check that $F^\circ \leq F^{\circ\circ}$. Given $r > s$ we have

$$\overline{F(r,-)} \leq \bigvee_{t>s} \overline{F(t,-)} = F^\circ(s,-),$$

hence $\overline{F(r,-)} = \overline{\overline{F(r,-)}} \leq \overline{F^\circ(s,-)}$. By recalling that $\{\overline{F(r,-)} : r \in \mathbb{Q}\}$ and $\{\overline{F^\circ(r,-)} : r \in \mathbb{Q}\}$ are scales that generate F° and $F^{\circ\circ}$, respectively, we get $F^\circ \leq F^{\circ\circ}$ according to Lemma 4.4(2).

(4) Let us calculate:

$$\begin{aligned}(F^\circ \wedge G^\circ)(r,-) &= \bigvee_{s>r} \overline{F(s,-)} \wedge \bigvee_{s>r} \overline{G(s,-)} \\ &\leq \bigvee_{s,t>r} \overline{F(s \wedge t,-) \wedge G(s \wedge t,-)} \\ &= \bigvee_{s>r} \overline{(F \wedge G)(s,-)} \\ &= (F \wedge G)^\circ(r,-),\end{aligned}$$

while the reverse inequality is obvious. □

As a corollary of Proposition 7.3 we have

$$\overline{\mathrm{LSC}}(L) = \{F \in \overline{\mathrm{F}}(L) : F = F^\circ\}$$

and

$$F^\circ = \bigvee \{G \in \overline{\mathrm{LSC}}(L) : G \leq F\}.$$

For the sake of completeness we include the dual variant of Proposition 7.3 (showing that the operator $(\cdot)^- : \overline{\mathrm{F}}(L) \to \overline{\mathrm{USC}}(L)$ behaves like a closure operator).

Proposition 7.4. *The following hold for all $F, G \in \overline{\mathrm{F}}(L)$:*

(1) $\perp^- = \perp$, *where* $\perp(r, -) = 1$ *for all* $r \in \mathbb{Q}$,
(2) $F \leq F^-$,
(3) $F^{--} = F^-$,
(4) $(F \vee G)^- = F^- \vee G^-$.

Also note that

$$\overline{\mathrm{USC}}(L) = \{F \in \overline{\mathrm{F}}(L) : F = F^-\}$$

and

$$F^- = \bigwedge\{G \in \overline{\mathrm{USC}}(L) : G \geq F\}.$$

Both $(\cdot)^{\circ -}$ and $(\cdot)^{-\circ}$ are idempotent, i.e. $F^{\circ - \circ -} = F^{\circ -}$ and $F^{-\circ -\circ} = F^{-\circ}$.

Now we are going to discuss the connections between the lower and upper regularizations in the sense of [9] with those introduced above. Given $g \in \mathrm{usc}(L)$, we put $\downarrow_{\mathrm{LSC}}(g) = \{f \in \mathrm{lsc}(L) : f \lhd g\}$ and let

$$\mathrm{usc}^\circ(L) = \{g \in \mathrm{usc}(L) : \downarrow_{\mathrm{LSC}}(g) \neq \varnothing\},$$
$$\mathrm{lsc}^-(L) = \{f \in \mathrm{lsc}(L) : \text{-}f \in \mathrm{usc}^\circ(L)\}.$$

For each $g, \text{-}f \in \mathrm{usc}^\circ(L)$, we define

$$g^\circ = \bigvee(\downarrow_{\mathrm{LSC}}(g)) \quad \text{and} \quad f^- = \text{-}(\text{-}f)^\circ.$$

Proposition 7.5. *The following hold:*

(1) $\Sigma(g^\circ) = \Upsilon(g)^\circ$ *for all* $g \in \mathrm{usc}^\circ(L)$;
(2) $\Upsilon(f^-) = \Sigma(f)^-$ *for all* $f \in \mathrm{lsc}^-(L)$.

Proof. To show (1), let $g \in \mathrm{usc}^\circ(L)$. By Propositions 6.1(1) and 6.2:

$$\Sigma(g^\circ) = \underset{\Sigma(h)\leq\Upsilon(g)}{\Sigma}\left(\bigvee_{\mathrm{lsc}(L)\ni h\lhd g} h\right) = \bigvee_{\Sigma(h)\leq\Upsilon(g)} \Sigma(h) = \Upsilon(g)^\circ.$$

As always, (2) follows from (1) by duality. □

Remark 7.6. In [9, Proposition 4.3] it is shown that, given $g, \text{-}f \in \mathrm{usc}^\circ(L)$, one has $g^\circ(r, -) = \bigvee_{s>r} g(-, s)^*$ and $f^-(-, r) = \bigvee_{s<r} f(s, -)^*$. These formulas make sense for arbitrary $g, \text{-}f \in \overline{\mathrm{usc}}(L)$ and Proposition 7.5 continues to hold in this more general setting.

Proposition 7.7. *The following hold:*

(1) $\Sigma(g^\circ) = \Upsilon(g)^\circ$ *for all* $g \in \overline{\mathrm{usc}}(L)$;
(2) $\Upsilon(f^-) = \Sigma(f)^-$ *for all* $f \in \overline{\mathrm{lsc}}(L)$.

Proof. For (1), since $g(-, s)^* \wedge g(-, s) = 0$, we get $\mathfrak{c}(g(-, s)^*) \leq \mathfrak{c}(g(-, s))^*$. Thus,

$$\mathfrak{c}(g^\circ(r, -)) = \mathfrak{c}\left(\bigvee_{s>r} g(-, s)^*\right) = \bigvee_{s>r} \mathfrak{c}(g(-, s)^*) \leq \bigvee_{s>r} \mathfrak{c}(g(-, s))^*.$$

Since $\mathfrak{c}(g^\circ(r, -))$ is closed, we get

$$\mathfrak{c}(g^\circ(r, -)) \leq \overline{\bigvee_{s>r} \mathfrak{c}(g(-, s))^*} = \overline{\Upsilon(g)(r, -)}.$$

The above inequality for scales gives $\Sigma(g^\circ) \leq \Upsilon(g)^\circ$. To get the reverse inequality we shall show that $\overline{\Upsilon(g)(r_1, -)} \leq \mathfrak{c}(g^\circ(r_2, -))$ whenever $r_1 > r_2$ (cf. Lemma 4.4). We have

$$\overline{\Upsilon(g)(r_1, -)} \leq \overline{\mathfrak{c}(g(-, r_1))^*} = \mathfrak{c}(g(-, r_1)^*) \leq \mathfrak{c}\left(\bigvee_{s>r_2} g(-, s)^*\right) = \mathfrak{c}(g^\circ(r_2, -)).$$

To have (2), given $f \in \overline{\mathrm{lsc}}(L)$, put $g = \text{-}f \in \overline{\mathrm{usc}}(L)$ into (1). □

Proposition 7.8. *Let $F \in \mathrm{F}(L)$. The following hold:*

(1) *If* $\bigvee_{r\in\mathbb{Q}} \overline{F(r, -)} = 1$, *then* $F^\circ \in \mathrm{LSC}(L)$;
(2) *If* $\bigvee_{r\in\mathbb{Q}} \overline{F(-, r)} = 1$, *then* $F^- \in \mathrm{USC}(L)$.

Proof. We only prove (1) (because (2) follows from (1) by the duality) by checking that (r5) and (r6) hold for the extended scale $\{\overline{F(r,-)} : r \in \mathbb{Q}\}$. For (r5) we have $\bigvee_{r\in\mathbb{Q}} F^\circ(r,-) = \bigvee_{r\in\mathbb{Q}}\bigvee_{r>s}\overline{F(s,-)} = \bigvee_{r\in\mathbb{Q}}\overline{F(r,-)} = 1$, while for (r6):

$$\bigvee_{r\in\mathbb{Q}} F^\circ(-,r) = \bigvee_{r\in\mathbb{Q}} (\overline{F(r,-)})^* \geq \bigvee_{r\in\mathbb{Q}} F(r,-)^* \geq \bigvee_{r\in\mathbb{Q}} F(-,r) = 1. \quad \square$$

We also note the following:

Corollary 7.9. *Let f, -$g \in \mathrm{lsc}(L)$. Then:*

(1) $f^- \in \mathrm{usc}(L)$ *if and only if* $\Sigma(f)^- \in \mathrm{USC}(L)$*;*
(2) $g^\circ \in \mathrm{lsc}(L)$ *if and only if* $\Upsilon(g)^\circ \in \mathrm{LSC}(L)$*.*

Proof. To show (1), let $f^- \in \mathrm{usc}(L)$. Then $\Sigma(f)^- = \Upsilon(f^-) \in \mathrm{USC}(L)$ by Proposition 6.1(3). The reverse implication follows similarly, while (2) is a consequence of (1) when applied to -g. $\square$

Properties 7.10. *For each complemented sublocale S of L the following hold: $(\chi_S)^- = \chi_{\overline{S}}$ and $(\chi_S)^\circ = \chi_{S^\circ}$. In particular, for each $a \in L$, $(\chi_{\mathfrak{c}(a)})^- = \chi_{\mathfrak{c}(a)}$, $(\chi_{\mathfrak{o}(a)})^- = \chi_{\mathfrak{c}(a^*)}$, $(\chi_{\mathfrak{o}(a)})^\circ = \chi_{\mathfrak{o}(a)}$ and $(\chi_{\mathfrak{c}(a)})^\circ = \chi_{\mathfrak{o}(a^*)}$.*

Appendix. Some insertion and extension theorems revisited

We close with a brief illustration of how the framework introduced here provides nice formulations of the known important insertion and extension theorems on semicontinuous real functions [8–10]. Up to now (cf. Introduction), lower and upper semicontinuous real functions had different domains. With certain abuse of notation (related to the symbol $\leq$), we had, for instance, written $f \leq h \leq g$ to denote the situation in which $f \in \mathrm{lsc}(L)$, $g \in \mathrm{usc}(L)$, and $h \in \mathrm{c}(L)$ were such that $f \blacktriangleleft g$, $f \leq h_{|\mathfrak{L}_l(\mathbb{R})}$, and $h_{|\mathfrak{L}_u(\mathbb{R})} \leq g$ (cf. [10]). Now, with F, G, and H being the images of f, g, and h under the embeddings Σ, Υ, and Ψ, respectively, we have just $F \leq H \leq G$ where all the three morphisms act on the same domain $\mathfrak{L}(\mathbb{R})$ and take values in the frame $\mathcal{S}(L)$ and $\leq$ denotes the partial order in $\mathrm{F}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{S}(L))$. The proofs of all the theorems which follow remain the same.

We start with the pointfree version of the Katětov–Tong insertion theorem which (after [16]) was the initial motivation for our research programme started with [17,13]. We need first to recall some terminology.

Let $\mathcal{D}_L = \{(a,b) \in L \times L : a \vee b = 1\}$. Then L is called *normal* if there exists a function $\Delta : \mathcal{D}_L \to L$ such that $a \vee \Delta(a,b) = 1 = b \vee \Delta(a,b)^*$ for all $(a,b) \in \mathcal{D}_L$. The operator Δ is called a *normality operator*.

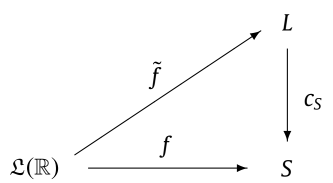
Theorem 8.1. *A frame L is normal if and only if, given an upper semicontinuous $G : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ and a lower semicontinuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ with $G \leq F$, there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $G \leq H \leq F$.*

Let $a, b \in L$ with $a \vee b = 1$. Then $\mathfrak{o}(b) \leq \mathfrak{c}(a)$. Therefore, $\chi_{\mathfrak{c}(a)} \leq \chi_{\mathfrak{o}(b)}$. Applying Theorem 8.1, there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\chi_{\mathfrak{c}(a)} \leq H \leq \chi_{\mathfrak{o}(b)}$. Hence (cf. [15]):

Corollary 8.2. *A frame L is normal if and only if for every $a, b \in \mathcal{D}_L$, there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\chi_{\mathfrak{c}(a)} \leq H \leq \chi_{\mathfrak{o}(b)}$.*

The existence of a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\chi_{\mathfrak{c}(a)} \leq H \leq \chi_{\mathfrak{o}(b)}$ means that there exists $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $h((-,0) \vee (1,-)) = 0$, $h(0,-) \leq a$ and $h(-,1) \leq b$. Thus, the corollary above is precisely the Urysohn's Lemma for frames [5] (cf. [2, Prop. 5]).

Let S be a sublocale of L and let $c_S : L \to S$ with $c_S(x) = \bigwedge\{s \in S : x \leq s\}$ be the corresponding frame quotient. We recall that an $f \in \mathrm{c}(S)$ has a *continuous extension* to L if there exists an $\tilde{f} \in \mathrm{c}(L)$ such that the following diagram commutes



i.e. $c_S \circ \tilde{f} = f$. We now say that $F \in \mathrm{C}(S)$ has a *continuous extension* to L if there exists an $\tilde{F} \in \mathrm{C}(L)$ such that $\tilde{F}(p,q) \vee S = F(p,q)$ for every $p, q \in \mathbb{Q}$. The next result provides the link between the old and the new approach to the extension problem.

Proposition 8.3. *Let S be a sublocale of L. Then $f \in \mathrm{c}(S)$ has a continuous extension to L if and only if $\mathfrak{c} \circ f$ has a continuous extension to L.*

Proof. The proof follows immediately from the fact that the closed sublocale $\mathfrak{c}(c_S(x))$ in S is precisely $\mathfrak{c}(x) \vee S$. $\square$

The pointfree variant of the Tietze extension theorem can now be stated as follows:

Theorem 8.4. *A frame L is normal if and only if for every closed sublocale S of L, each continuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(S)$ has a continuous extension $\tilde{F} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\tilde{F}(p, q) \vee S = F(p, q)$ for every $p, q \in \mathbb{Q}$.*

We now move to insertion and extension theorems for monotonically normal frames. Equip $\mathcal{D}_L = \{(a, b) \in L \times L : a \vee b = 1\}$ with the componentwise order inherited from $L^{op} \times L$. Then L is called *monotonically normal* if there exists a monotone normality operator. Further, the set

$$\mathrm{UL}(L) = \{(G, F) \in \mathrm{USC}(L) \times \mathrm{LSC}(L) : G \le F\}$$

carries the componentwise order induced from $\mathrm{F}(L)^{op} \times \mathrm{F}(L)$. The following comes from [8, Theorem 5.4]. Even if the statements look similar, this is a good place to repeat again how advantageous is the approach of considering $\mathcal{S}(L)$-valued morphisms. The reader should consult [8] to see how much effort is saved by moving from $\mathrm{usc}(L) \times \mathrm{lsc}(L)$ to $\mathrm{USC}(L) \times \mathrm{LSC}(L)$.

Theorem 8.5. *A frame L is monotonically normal if and only if there exists a monotone function $\Lambda : \mathrm{UL}(L) \to \mathrm{C}(L)$ such that $G \le \Lambda(G, F) \le F$ for all $(G, F) \in \mathrm{UL}(L)$.*

Given a sublocale S of L, a function $\Phi : \mathrm{C}(S) \to \mathrm{C}(L)$ is called an *extender* if $\Phi(F)$ extends F for all $F \in \mathrm{C}(S)$. Let S be a closed sublocale of L and $F \in \mathrm{C}(S)$. We define $F^l \in \mathrm{LSC}(L)$ and $F^u \in \mathrm{USC}(L)$ as follows:

$$F^l(r, -) = \begin{cases} 1 & \text{if } r < 0 \\ F(r, -) & \text{if } 0 \le r < 1 \\ 0 & \text{if } r \ge 1 \end{cases} \quad \text{and} \quad F^l(-, r) = \begin{cases} 0 & \text{if } r \le 0 \\ \bigvee_{s<r} F(s, -)^* & \text{if } 0 < r \le 1 \\ 1 & \text{if } r > 1, \end{cases}$$

$$F^u(r, -) = \begin{cases} 1 & \text{if } r < 0 \\ \bigvee_{s>r} F(-, s)^* & \text{if } 0 \le r < 1 \\ 0 & \text{if } r \ge 1 \end{cases} \quad \text{and} \quad F^u(-, r) = \begin{cases} 0 & \text{if } r \le 0 \\ F(-, r) & \text{if } 0 < r \le 1 \\ 1 & \text{if } r > 1. \end{cases}$$

It is easy to check that $F^u \le F^l$, i.e. $(F^u, F^l) \in \mathrm{UL}(L)$. The following is the reformulation of [8, Theorem 6.4] in our new setting:

Theorem 8.6. *For L a frame the following are equivalent:*

(1) *L is monotonically normal;*
(2) *For each closed sublocale S of L there exists an extender $\Phi : \mathrm{C}(S) \to \mathrm{C}(L)$ such that for every closed sublocales S_1 and S_2 of L and $F_i \in \mathrm{C}(S_i)$ $(i = 1, 2)$ with $({F_1}^u, {F_1}^l) \le ({F_2}^u, {F_2}^l)$ one has $\Phi(F_1) \le \Phi(F_2)$.*

Recall that a frame L is called *extremally disconnected* if $a^* \vee a^{**} = 1$ for all $a \in L$. We have the following (cf. [9]):

Theorem 8.7. *For L a frame the following are equivalent:*

(1) *L is extremally disconnected;*
(2) *Given lower semicontinuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ and upper semicontinuous $G : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ with $F \le G$, there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $F \le H \le G$;*
(3) *For every open sublocale S of L, each continuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(S)$ has a continuous extension $\tilde{F} : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\tilde{F}(p, q) \vee S = F(p, q)$ for every $p, q \in \mathbb{Q}$.*

Let $F \in \mathrm{F}(L)$. We write $F \ge \mathbf{0}$ if $F(-, 0) = 0$. Similarly, $F \le \mathbf{1}$ means that $F(1, -) = 0$. Also, we write $F > \mathbf{0}$ whenever $F(0, -) = 1$. The following two results come from [10].

We recall that a frame L is *perfectly normal* if for each $a \in L$ there is a countable subset $B \subseteq L$ such that $a = \bigvee B$ and $a \vee b^* = 1$ for all $b \in B$.

Theorem 8.8. *For L a frame the following are equivalent:*

(1) *L is perfectly normal;*
(2) *L is normal and for each lower semicontinuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ with $\mathbf{0} \le F \le \mathbf{1}$ there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\mathbf{0} < H \le F$ and $H(0, -) = F(0, -)$;*
(3) *For every closed sublocale S of L, each continuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(S)$ with $\mathbf{0} \le F \le \mathbf{1}$ has a continuous extension $\tilde{F} \in \mathrm{C}(L)$ such that $\tilde{F}(0, 1) \ge S$.*

Finally, recall that a frame L is *countably paracompact* [6] if for every subset $\{a_n : n \in \mathbb{N}\} \subseteq L$ with $\bigvee_n a_n = 1$ there exists a subset $\{b_n : n \in \mathbb{N}\} \subseteq L$ such that $\bigvee_n b_n = 1$ and $a_n \vee b_n^* = 1$ for all n. In the last example we restate from [10] the following:

Theorem 8.9. *For a normal frame L, the following are equivalent:*

(1) *L is countably paracompact;*
(2) *For each lower semicontinuous $F : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ satisfying $\mathbf{0} < F \le \mathbf{1}$ there exists a continuous $H : \mathfrak{L}(\mathbb{R}) \to \mathcal{S}(L)$ such that $\mathbf{0} < H < F$.*

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