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UNIVERSIDADE D
COIMBRA

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On lax factorisation systems

Tese no âmbito do Programa Interuniversitário de Doutoramento em Matemática, orientada pela Professora Doutora Maria Manuel Clementino e apresentada ao Departamento de Matemática da Faculdade de Ciências e Tecnologia da Universidade de Coimbra

Abril de 2022

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UC|UP Joint PhD Program in Mathematics

Programa Interuniversitário de Doutoramento em Matemática

PhD Thesis | Tese de Doutoramento

May 2022

Acknowledgements

This is the closure of a journey that took me four years to walk and that had a considerable impact on my life starting from the most evident fact that I got to live far from home in Portugal. For this reason I will need a few lines to thank the people that crossed my path in these years of difficulties, joy, learning and change.

First of all I would like to thank my family for the love and support that they have always shown and provided selflessly.

I shall also thank my supervisor Maria Manuel Clementino for dedicating her time, patience and help to me and my work along this path. I valued and still value her guidance and input more than I will ever be able to demonstrate.

Moreover, I would like to gratefully acknowledge the precious insights and corrections to the thesis of the members of the jury: John Bourke, Graham Manuell, Jorge Picado, Diana Rodelo, Lurdes Sousa.

Then I would like to thank the people that helped me at the beginning of this journey, pointing me the way with suggestions and unvaluable teachings in particular prof. Alessandro Ardizzoni, prof. Marino Gran and Paolo Saracco.

I would like to dedicate this work to the dear memory of Prof. Alberto Collino, who was the first person to introduce me to categories.

I shall also dedicate this work and achievement to the people that despite four years of living geographically far, never failed to be a caring and friendly presence in my life. Those who have never missed a day in my life and have always been there for engaging constructive discourses, offering friendly support during hard times and sharing cheerful celebrations in the moments of joy. Thank you Adriano Barile, Junior Perri, Nietta Ricciardi and Giovanni Russo. To Giovanni in particular I owe a lot for his help and insight during the writing of this thesis.

I shall include in my thanks all the people that were there for me whenever I passed from Italy and that kept being a certainty during four long years of distance like Monica Barbero, Simone Dutto, Marzia Garzetti, Lorenzo Lavagna, Alessandra Macciotta and Stefano Rovea, Luisa Macciotta, Alessandro Mele, Gabriele Midolo, Maria Motta, Andrea Orizzonte, Maria Sergio, Cinzia D'Alessandro, Pino Rossetti, Alberto Rossetti. I would also like to mention the friends around the world that helped me get through the COVID-19 pandemic with company and affection: Valeria Tacchi, Ilona Koskela, Chiara Balestra, Lorenza Galeota, Alfonso Sensale and Andrea Solimeno.

Finally I should thank Alessandra Teresa Cignarella, who gave me the strength and support to start this journey.

These four years have also meant finding new friends and family. The first and the most joyful new entry is the birth of Jack Perri.

Then I cannot forget to mention the fortunate encounter with Oskar Weinberger, who shared with me the joys and struggles of our first year in Porto, and the other friends from Porto: Helöise Andris, Atefeh Afsar, Nikos Tsopanidis, Peter Lombaers, Elisa Filippi, Alessandra Telesca, Hazal Yavuz. Finally I would like to mention Bruno Pires, my long time flatmate in Coimbra.

I gratefully acknowledge the Centre for Mathematics of the University of Coimbra and the Mathematic Department for hosting me and giving me the opportunity to meet unvaluable researchers. In particular I would like to aknowledge the collegues that I had the chance to meet during this years: Igor Arrieta, Ana Belén Avilez García, Herman Goulet-Ouellet, David Jesus, Fosco Loregian, Ayk Telciyan, Anna Laura Suarez.

Then last, but not at all least, I would like to remember and thank from the depth of my hearth GEFAC, the group of friends which has welcomed me and turned the dark times, isolation and discourage that I was experiencing into friendly and fulfilling moments that I will hold dear forever. In particular I shall thank Diogo Lobo for introducing me to GEFAC; the Diréssão géfaque 20/21: Amabile Bezinelli, Pedro Campos, Camila Dias, Maria Dias, Simão Lopes, Mariana M. Morais; and moreover, among the others, Rita Bràs, Tiago Ferreira, Rita Gonçalves, Mariana Keating, Hannah Kuhn, Patrícia Mendonça, João Monteiro, Carolina Oliveira, Saulo Oliveira, Carolina Rocha, João Santos and Claudia Tirone.

In conclusion I would like to thank a person that I have never met, but whose readings changed my way of looking at the world and helped me build a better, healthier and more conscious perspective on what to do with my life. Thank you Raffaele Alberto Ventura.

Financial support The author gratefully acknowledge the financial support from the Centre for Mathematics of the University of Coimbra (UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES). The author also acknowledges the support of a PhD grant from FCT/MCTES (PD/BD/142958/2018).

Abstract

The main focus of this thesis are factorisation systems and their applications to categories of partial maps. The original contribution of this work is the introduction of a new type of factorisation systems developed in the context of categories enriched over the category of partial orders and a presentation of various instances and constructions of such structures in the context of categories of partial maps.

The first part of the thesis constitutes a survey on the traditional literature about different types of factorisation systems. These structures are presented starting from their definition together with their main properties and some of the most known results on the subject.

The second is an introduction to categories of partial maps which is intended to be an helpful source of tools for the reader that approaches the results in the following chapters.

Then we proceed to present the original contribution introducing the notion of lax weak orthogonality, which involves the existence of diagonal morphisms for lax squares. Inspired by the traditional theory of factorisation systems from the second chapter, we proceed to introduce the definitions of lax weak factorisation systems, lax functorial factorisation systems and lax algebraic weak factorisation systems. We focus on the study of their main features and properties, in particular we investigate the links between these concepts. We conclude by observing that the arguments developed have a dual formulation that concerns oplax squares.

The last part is dedicated to the application of the newly defined structures in the context of partial maps. We consider categories of partial maps enriched over the category of partial orders. Then we show that any category of partial maps comes equipped with a lax algebraic weak factorisation system, which isolates the domain component from the total datum of a partial map.

Furthermore, we explore the close link between oplax weak factorisation systems on a category of partial maps and the oplax weak factorisation systems on the base category which carry some conditions of stability under pullbacks. In fact, we will succeed in establishing a bijection between the two classes. Moreover, we will show that functoriality and monad structures are transferred from orthogonal factorisation systems on the base category to those induced among partial maps when considering the simplest notion of Ord-enrichment.

We conclude presenting some remarks on cofibrant generation of lax and oplax weak factorisation systems for certain pointed categories.

Resumo

O foco principal desta tese são os sistemas de fatorização e as suas aplicações às categorias de funções parciais. A contribuição original deste trabalho é a introdução de um novo tipo de sistemas de factorização desenvolvidos no contexto de categorias enriquecidas na categoria dos espaços ordenados e uma apresentação de várias instâncias e construções de tais estruturas no contexto das categorias de funções parciais.

A primeira parte da tese consiste numa apresentação dos resultados clássicos sobre diferentes tipos de sistemas de fatorização. Essas estruturas são apresentadas a partir das próprias definições juntamente com as principais propriedades e alguns dos resultados mais conhecidos sobre o assunto.

A segunda é uma introdução às categorias de funções parciais que se destina a ser uma fonte útil de ferramentas para o leitor na abordagem dos resultados nos capítulos seguintes.

Seguindo, passamos a apresentar a contribuição original introduzindo a nova noção de ortogonalidade fraca lassa, que envolve a existência de morfismos diagonais para quadrados lassos. Inspirado pela teoria tradicional dos sistemas de fatorização do primeiro capítulo, apresentamos a definição de sistemas de fatorização fracos lassos, sistemas de fatorização functorial lassos e sistemas de fatorização fracos algébricos lassos. Concentramo-nos no estudo das principais características e propriedades, em particular investigamos as ligações entre esses conceitos. Concluímos observando que os argumentos desenvolvidos têm uma formulação dual ao considerar os quadrados oplassos que produz um conjunto igualmente poderoso de resultados.

A última parte é dedicada à aplicação das novas estruturas definidas no contexto de funções parciais. Consideramos categorias de funções parciais enriquecidas na categoria dos espaços ordenados. Seguindo, mostramos que qualquer categoria de funções parciais vem equipada com um sistema algébrico de fatorização lasso, que isola o componente do domínio do datum total de uma função parcial.

Em seguida, exploramos a estreita ligação entre os sistemas de fatorização fracos oplassos de uma categoria de funções parciais e os sistemas de fatorização fracos oplasso na categoria de base, que apresentam algumas condições de estabilidade sob produtos fibrados. Na verdade, conseguiremos estabelecer uma bijeção entre as duas classes de fatorizações. Além disso, mostraremos que a functorialidade e as estruturas da monada são transferidas dos sistemas de fatorização ortogonais na categoria de base para aqueles induzidos entre as funções parciais ao considerar a noção mais simples de enriquecimento em Ord.

Concluímos apresentando algumas observações sobre a geração cofibrante de sistemas de fatorização lassos e oplassos para certas categorias pontuadas.

Contents

Chapter 1

Introduction

Factorisation systems are a categorical concept that have been around for quite some time among the categorical structures. They have proved to be a handy tool to analyse the structure of a category, entering with good reasons in the categorical benchmark of many category theorists.

The first introduction of factorisation systems was in [FK72] and these particular factorisations were later called *orthogonal factorisation systems*, to distinguish them from the many other structures that in time joined the rich folder of factorisations of morphisms in a category.

This is the topic from which this investigation started and in particular from the study of factorisation systems that had an interplay with the higher structures of a category introduced in [CLF16, LF19, CLF20]. In this research our objective has been to capture and describe the properties of a new orthogonality relation that gives rise to lifting properties for a wider class of laxly commutative squares and that was endowed with the structure of **Ord**-enrichment. This new orthogonality relation, that we called *lax weak orthogonality*, allowed us to expand the theory of factorisation systems in this sense introducing newer structures that mirror the classical theory for factorisation systems. In particular we discuss a functorial approach to lax weak orthogonal factorisations and explored its features and properties, then we present how these can be equipped with lax monad and comonad structures as already happens for the classical *algebraic weak factorisation systems*.

Then our attention turned to categories of partial maps. These categories are an interesting environment for **Ord**-categorical studies and, moreover, have a vast field of applications as witnessed by the large number of studies that make use of this mathematical tool, ranging from analysis and topology to semigroup theory and computer science.

During our study, categories of partial maps have proved to be quite a fruitful context to develop lax and oplax factorisation systems and, as we will see, some of them are actually quite hardly encoded in the structure of such categories.

We provide now a quick plan of the work. In Chapter 2 we discuss an overview on the literature regarding factorisation systems. The structures are presented in order of generality and accompanied by their properties and features. We go through the constructions that had a bigger correlation with the original content of the thesis. We conclude the chapter discussing some other studies that, similarly to our endeavour, investigated the interplay between factorisations and higher categorical structures.

In Chapter 3 we focus on the categorical approach to partial maps, providing a brief survey on categories of partial maps, their study and application. Then we move on to discuss their formalisation and some results that will come in handy for the purposes of our work.

In the next two chapters we present the original contribute of this thesis, which regards lax factorisation systems and their applications; this material was developed in [Lar21].

In Chapter 4 we present lax weak orthogonality discussing its features and then we proceed to introduce lax weak factorisation systems, lax functorial factorisation systems and lax algebraic weak factorisation systems. In doing so, we discuss a small object argument for cofibrant generation of lax weak factorisation systems and we study the correlation between lax weak factorisation systems and lax functorial factorisation systems.

Finally, Chapter 5 is dedicated to the development of examples of lax factorisation systems in the context of categories of partial maps. First we prove the existence, for each category of partial maps, of a lax algebraic weak factorisation system that separates the domain component and the total datum of a partial morphism. Then we discuss how oplax weak factorisation systems on the base category, under some pullback stability conditions, give rise to oplax weak factorisations among partial maps and vice versa. Moreover, we show that this gives rise to a bijection between the two classes of factorisation systems. Furthermore, we prove that for particular **Ord**-enrichments the functorial properties of an orthogonal factorisation system are transferred from the base category to the category of partial maps. We conclude presenting some remarks on cofibrant constructions of lax and oplax weak factorisation systems for certain pointed **Ord**-categories along to some applications to categories of partial maps.

Chapter 2

Factorisation systems

The following chapter is intended as an overview on factorisation systems, providing the context and framework of our study. The concepts, definitions and results presented along this chapter are already part of an established scientific literature and we will provide references along the discussion.

It is worth to mention that historically the first notion of factorisation system was introduced by Freyd and Kelly in [FK72]. We organized this chapter not by historical order of appearance, but in order of generality, starting from the weaker notions, going on to refine them and describing stronger structures through the discussion. For this reason the first structures mentioned will be the relatively newer *weak factorisation systems* and *orthogonal factorisation systems* will be the penultimate.

We will present in the conclusion of the chapter the definition of *lax orthogonal factorisation systems*, and *enriched factorisation systems*, which are other kinds of factorisation systems linked to enrichments, as the ones that are the scope of this study.

2.1 Weak Factorisation Systems

This section is dedicated to present the most general notion of factorisation systems. We can trace back the introduction of these factorisation systems to Quillen's work in [Qui67]. Weak factorisation systems are an important element in the theory of Quillen model categories, in fact each model structure in Quillen's definitions entails two weak factorisation systems, as denoted in Definition 2.13. The results discussed in this section are known and they can be found in [AHRT02, RT02, Rie11].

We will denote commutative squares of morphisms in a given category $\mathscr{C}$ as

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

by $[u,v]: f \longrightarrow g$. This notation portrays commutative squares as morphisms in the arrow category $\mathscr{C}^2$.

Definition 2.1. *A morphism l is* ***left weakly orthogonal*** *to a morphism r, denoted by $l \boxslash r$ (sometimes in literature also denoted by $l \pitchfork r$), if for every commutative square $[u,v]: l \longrightarrow r$ there exists a morphism*

δ as in the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "l"'] & B \arrow[d, "r"] \\ C \arrow[r, "v"'] \arrow[ur, dashed, "\delta" description] & D, \end{tikzcd}$$

such that the two triangles commute. The morphism δ is said to be a ***diagonal morphism*** *or diagonal lifting of l against r for the square $[u,v]$.*

Given a class $\mathcal{H}$ of morphisms in $\mathscr{C}$, we can define the weak orthogonal complements of $\mathcal{H}$ as follows

$$^{\boxslash}\mathcal{H} = \{f | f \boxslash h \text{ for every } h \in \mathcal{H}\} \qquad \text{and} \qquad \mathcal{H}^{\boxslash} = \{f | h \boxslash f \text{ for every } h \in \mathcal{H}\}.$$

Proposition 2.2. *The pair $^{\boxslash}(-)$ and $(-)^{\boxslash}$ forms a Galois connection among the classes of morphisms in $\mathscr{C}$ partially ordered by the inclusion.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms in $\mathscr{C}$. We prove first that $^{\boxslash}(-)$ and $(-)^{\boxslash}$ are antitone with respect to inclusion. Let $\mathcal{A} \subseteq \mathcal{B}$. If $f \in {}^{\boxslash}\mathcal{B}$, then $f \boxslash b$ for every $b \in \mathcal{B}$, then in particular $f \boxslash a$ for every $a \in \mathcal{A}$, then $f \in {}^{\boxslash}\mathcal{A}$. Therefore $^{\boxslash}\mathcal{A} \supseteq {}^{\boxslash}\mathcal{B}$. Similarly one proves that $\mathcal{A}^{\boxslash} \supseteq \mathcal{B}^{\boxslash}$.

Moreover, $\mathcal{A} \subseteq {}^{\boxslash}\mathcal{B}$ if and only if for every $a \in \mathcal{A}$ and for every $b \in \mathcal{B}$, $a \boxslash b$, which is equivalent to $\mathcal{B} \subseteq {}^{\boxslash}\mathcal{A}$. Analogously $\mathcal{A} \subseteq \mathcal{B}^{\boxslash}$ is equivalent to $\mathcal{B} \subseteq \mathcal{A}^{\boxslash}$. □

Applying the previous results on inclusions of weak orthogonal complements we deduce the following corollary.

Corollary 2.3. *Let $\mathcal{H}$ be a class of morphisms in $\mathscr{C}$. Then $\mathcal{H} \subseteq {}^{\boxslash}(\mathcal{H}^{\boxslash})$ and $\mathcal{H} \subseteq ({}^{\boxslash}\mathcal{H})^{\boxslash}$. Moreover, $\mathcal{H}^{\boxslash} = ({}^{\boxslash}(\mathcal{H}^{\boxslash}))^{\boxslash}$ and $^{\boxslash}\mathcal{H} = {}^{\boxslash}\left(({}^{\boxslash}\mathcal{H})^{\boxslash}\right)$.*

We prove the following lemma.

Lemma 2.4. *Let f be a morphism in $\mathscr{C}$. Then the following are equivalent:*

1. *$f \boxslash g$ for every morphism g in $\mathscr{C}$;*
2. *$g \boxslash f$ for every morphism g in $\mathscr{C}$;*
3. *$f \boxslash f$;*
4. *f is an isomorphism.*

Proof. 1.∨2.⇒3. Straightforward.

3.⇒4. If f is weakly orthogonal to itself, then we have diagonal lifting for the square

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "\delta" description] & B; \end{tikzcd}$$

therefore the commutativity of the two triangles yields that f is an isomorphism.

3.⇒1. We consider a commutative square $[u,v] : f \longrightarrow g$. Then it is trivial to check that $u \cdot f^{-1}$ is a diagonal lifting for the square $[u,v]$.

3.⇒2. Is analogous to the proof of the last implication.

4.⇒3. It holds trivially, since one can build diagonal liftings through f^{-1}. □

Proposition 2.5. *Let $\mathcal{H}$ be a class of morphisms of $\mathscr{C}$. Then the following assertions hold.*

1. *$\mathcal{H}^{\boxslash}$ and ${}^{\boxslash}\mathcal{H}$ contain all the isomorphisms of $\mathscr{C}$;*

2. *$\mathcal{H}^{\boxslash}$ and ${}^{\boxslash}\mathcal{H}$ are closed under composition.*

Proof. 1. This is a consequence of Lemma 2.4.

2. We show this by building subsequent diagonal liftings. Let $f, g \in \mathcal{H}^{\boxslash}$ be two composable arrows. Then for every commutative square $[u,v] : h \longrightarrow g \cdot f$, with $h \in \mathcal{H}$, we can build the following diagram

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[dd, "h"'] & \bullet \arrow[d, "f"] \\
 & \bullet \arrow[d, "g"] \\
\bullet \arrow[r, "v"'] \arrow[uur, dotted, "\delta_{g\cdot f}"] \arrow[ur, dotted, "\delta_g"] & \bullet;
\end{tikzcd}$$

where $\delta_{g\cdot f}$ is a diagonal lifting for the square $[u, \delta_g]$. Then we have

$$g \cdot f \cdot \delta_{g\cdot f} = g \cdot \delta_g = v$$

since the lower triangle is commutative by construction. We conclude that $\delta_{g\cdot f}$ is a diagonal lifting for $[u,v]$. Then $g \cdot f \in \mathcal{H}^{\boxslash}$. The proof for ${}^{\boxslash}\mathcal{H}$ is similar. □

We recall the definition of binary products in the category $\mathscr{C}^2$.

Definition 2.6. *Let $\mathscr{C}$ be a category with binary products. Let f, g be two objects of $\mathscr{C}^2$. Then $f \times g$ is the universal arrow related to the product $B \times B'$ with respect to the span $(f \cdot \pi_A; g \cdot \pi_{A'})$, as depicted in the following diagram*

$$\begin{tikzcd}
 & A \times A' \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_{A'}"] \arrow[dd, dotted, "f\times g"] & \\
A \arrow[d, "f"'] & & A' \arrow[d, "g"] \\
B & B \times B'. \arrow[l, "\pi_B"] \arrow[r, "\pi_{B'}"'] & B'
\end{tikzcd}$$

Proposition 2.7. *Let $\mathcal{H}$ be a class of morphisms of a category $\mathscr{C}$ with products and pullbacks. The class $\mathcal{H}^{\boxslash}$ is closed under pullbacks and products in $\mathscr{C}^2$. Moreover, if e is a retraction and $f \cdot e \in \mathcal{H}^{\boxslash}$, then $f \in \mathcal{H}^{\boxslash}$.*

Proof. Let us consider $f \in \mathcal{H}^{\boxslash}$ and $g^*(f)$ its pullback along any morphism g. For every commutative square $[u,v] : h \longrightarrow g^*(f)$, with $h \in \mathcal{H}$, we have the following diagram

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, "h"'] & \bullet \arrow[r, "f^*(g)"] \arrow[d, "g^*(f)"] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "v"'] \arrow[urr, dotted, "\delta"] & \bullet \arrow[r, "g"'] & \bullet, \end{tikzcd}$$

where δ is the diagonal morphism lifting h against f. Then there exists a universal morphism δ' for the diagram

$$\begin{tikzcd} \bullet \arrow[drr, bend left, "\delta"] \arrow[dr, dotted, "\delta'"] \arrow[ddr, bend right, "v"'] & & \\ & \bullet \arrow[r, "f^*(g)"] \arrow[d, "g^*(f)"] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, "f"] \\ & \bullet \arrow[r, "g"'] & \bullet. \end{tikzcd} \tag{2.i}$$

We have the following equalities

$$\begin{cases} f^*(g) \cdot \delta' \cdot h = \delta \cdot h = f^*(g) \cdot u \\ g^*(f) \cdot \delta' \cdot h = v \cdot h = g^*(f) \cdot u, \end{cases}$$

and since pullback morphisms are jointly monomorphic, then $\delta' \cdot h = u$. By (2.i) we have that $g^*(f) \cdot \delta' = v$. This yields that δ' is a diagonal morphism lifting h against $g^*(f)$ and therefore $g^*(f) \in \mathcal{H}^{\boxslash}$.

Then we prove that $\mathcal{H}^{\boxslash}$ is closed under products in $\mathscr{C}^2$. Let us consider a family of morphisms $(f_i : A_i \longrightarrow B_i)_{i\in I}$ in $\mathcal{H}^{\boxslash}$ and $f = \prod_{i\in I} f_i : \prod_{i\in I} A_i \longrightarrow \prod_{i\in I} B_i$. We aim to prove that $f \in \mathcal{H}^{\boxslash}$. Let $[u,v] : h \longrightarrow f$ be a commutative square. For every $i \in I$ there exists δ_i lifting h against f_i as in the diagram

$$\begin{tikzcd} H \arrow[r, "u"] \arrow[d, "h"'] & \prod_{i\in I} A_i \arrow[r, "\pi_i"] \arrow[d, "f"] & A_i \arrow[d, "f_i"] \\ K \arrow[r, "v"'] \arrow[ur, dashed, "\langle\delta_i\rangle"] \arrow[urr, dotted, "\delta_i"] & \prod_{i\in I} B_i \arrow[r, "\pi_i'"'] & B_i. \end{tikzcd} \tag{2.ii}$$

Then there exists the universal arrow $\langle\delta_i\rangle$ for the product $\prod_{i\in I} A_i$. Since $(\pi_i)_{i\in I}$ and $(\pi_i')_{i\in I}$ are jointly monomorphic, we have that

$$\begin{cases} \pi_i \cdot u = \delta_i \cdot h = \pi_i \cdot \langle\delta_i\rangle \cdot h \\ \pi_i' \cdot v = f_i \cdot \delta_i = f_i \cdot \pi_i \cdot \langle\delta_i\rangle = \pi_i' \cdot f \cdot \langle\delta_i\rangle \end{cases} \Rightarrow \begin{cases} u = \langle\delta_i\rangle \cdot h \\ v = f \cdot \langle\delta_i\rangle. \end{cases}$$

This concludes that $f \in \mathcal{H}^{\boxslash}$.

Now let us consider $f\cdot e\in\mathcal{H}^{\boxslash}$, with $e\cdot m=\mathrm{id}$. If $[u,v]:h\longrightarrow f$ is any commutative square with $h\in\mathcal{H}$, then there exists a diagonal morphism in the following commutative diagram

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[dd, "h"'] & \bullet \arrow[r, "m"] & \bullet \arrow[d, "e"] \\
 & & \bullet \arrow[d, "f"] \\
\bullet \arrow[rr, "v"'] \arrow[uurr, dotted, "d"] & & \bullet .
\end{tikzcd}$$

We consider $\delta=e\cdot d$. We deduce that δ is a diagonal lifting, since

$$\begin{cases} u=e\cdot m\cdot u=e\cdot d\cdot h=\delta\cdot h \\ v=f\cdot e\cdot d=f\cdot\delta. \end{cases}$$

$\square$

We remark that the previous proposition induces the following dual result.

Proposition 2.8. *Let $\mathcal{H}$ be a class of morphisms of a category $\mathscr{C}$ with coproducts and pushouts. The class ${}^{\boxslash}\mathcal{H}$ is closed under pushouts and coproducts in $\mathscr{C}^2$. Moreover, if m is a coretraction and $m\cdot f\in{}^{\boxslash}\mathcal{H}$, then $f\in{}^{\boxslash}\mathcal{H}$.*

Definition 2.9. *A **weak prefactorisation system** is a pair of classes of morphisms $(\mathcal{L},\mathcal{R})$ such that $\mathcal{L}^{\boxslash}=\mathcal{R}$ and ${}^{\boxslash}\mathcal{R}=\mathcal{L}$. If every morphism f admits an $(\mathcal{L},\mathcal{R})$-factorisation*

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "\mathcal{L}\ni l"'] & & B \\
 & M_f, \arrow[ur, "r\in\mathcal{R}"'] &
\end{tikzcd}$$

*then $(\mathcal{L},\mathcal{R})$ is called **weak factorisation system** (WFS).*

There exists an equivalent definition of WFS. In the following proposition we will state it and prove its equivalence to the previous definition.

Proposition 2.10. *The pair $(\mathcal{L},\mathcal{R})$ is a WFS if and only if the following conditions hold:*

1. *$\mathcal{L}\boxslash\mathcal{R}$, i.e. for every $l\in\mathcal{L}$ and $r\in\mathcal{R}$, then $l\boxslash r$;*
2. *every morphism f admits an $(\mathcal{L},\mathcal{R})$-factorisation;*
3. *if $\mathrm{id}=e\cdot m$, then*
   (a) if $m\cdot f\in\mathcal{L}$, then $f\in\mathcal{L}$;
   (b) if $g\cdot e\in\mathcal{R}$, then $g\in\mathcal{R}$.

Proof. $\Rightarrow$. It is straightforward that if $(\mathcal{L},\mathcal{R})$ is a WFS, then *1.* and *2.* are true and *3.* is a consequence of Propositions 2.7 and 2.8.

$\Leftarrow$. We aim to prove that $\mathcal{L}$ and $\mathcal{R}$ are weak orthogonal complements with respect to each other. Condition *1.* implies that $\mathcal{L} \subseteq {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} \subseteq \mathcal{L}^{\boxslash}$. Let $f \in {}^{\boxslash}\mathcal{R}$. By condition *2.* we have the factorisation $f = r_f \cdot l_f$. Hence the existence of a diagonal lifting in the following diagram

$$\begin{tikzcd} A \arrow[r, "l_f"] \arrow[d, "f"'] & M_f \arrow[d, "r_f"] \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, dotted, "\delta"] & B. \end{tikzcd}$$

Then δ is a coretract by commutativity of the lower triangle and $\delta \cdot f = l_f \in \mathcal{L}$. Therefore $f \in \mathcal{L}$ by *3.(a)*. Hence $\mathcal{L} = {}^{\boxslash}\mathcal{R}$. The proof that $\mathcal{R} = \mathcal{L}^{\boxslash}$ is analogous. □

Example 2.11. *We provide a first example for* **Set**, *which is a sort of guiding example for* WFSs. *Let us consider the class of epimorphisms* Epi *and the class of monomorphisms* Mono. *Then the pair* (Mono, Epi) *is a* WFS. *Recalling Proposition 2.10, we notice that condition 3. is trivially satisfied by the two classes. We consider a square* $[u,v] : m \longrightarrow e$, *where* $m \in$ Mono *and* $e \in$ Epi. *Then we can always build a diagonal morphism as*

$$\delta(x) = \begin{cases} u\left(m^{-1}(x)\right) & \text{if } x \in \mathrm{Im}(m) \\ y \in e^{-1}(v(x)) \neq \emptyset & \text{otherwise.} \end{cases}$$

Thereafter we have that each morphism $f : A \longrightarrow B$ *can be factorised as*

$$A \xrightarrow{\langle \mathrm{id}_A, f\rangle} A \times B \xrightarrow{\pi_B} B \qquad \text{if } A \neq \emptyset$$

$$\emptyset \longrightarrow B \xrightarrow{\mathrm{id}_B} B \qquad \text{otherwise.}$$

Moreover, it admits also another distinct (Mono, Epi)*-factorisation*

$$A \longrightarrow A \amalg B \longrightarrow B.$$

Hence (Mono, Epi) *is a* WFS.

Example 2.12. *As cited in [Rie14], we actually have that on* **Set** *there exist only the following* WFSs*:*

$$\bullet\,(\mathsf{All}, \mathsf{Iso}) \quad \bullet\,(\mathsf{Epi}, \mathsf{Mono}) \quad \bullet\,(\mathsf{All} \backslash \mathcal{N}, \mathsf{Iso} \cup \mathcal{N})$$
$$\bullet\,(\mathsf{Iso}, \mathsf{All}) \quad \bullet\,(\mathsf{Mono}, \mathsf{Epi}) \quad \bullet\,(\mathsf{Mono} \backslash \mathcal{N}, \mathsf{Epi} \cup \mathcal{N}),$$

where $\mathcal{N} = \{f : \emptyset \longrightarrow B | B \neq \emptyset\}$.

In conclusion we report the definition of model structures, which highlights the link with WFSs.

Definition 2.13. *A* ***model structure*** *on a category* $\mathscr{C}$ *is constituted by three classes of morphisms*

- Cof *the class of cofibrations;*

- Fib *the class of fibrations;*
- $\mathcal{W}$ *the class of weak equivalences.*

These classes must satisfy the following:

1. $\mathcal{W}$ *contains all isomorphism;*
2. *considering f, g, $g \cdot f$, then if two of them belong to $\mathcal{W}$, the third one belongs to $\mathcal{W}$ as well (referred to as the two out of three condition);*
3. $(\mathsf{Cof}, \mathsf{Fib} \cap \mathcal{W})$ *and* $(\mathsf{Cof} \cap \mathcal{W}, \mathsf{Fib})$ *are both* WFS*s.*

2.2 Small object argument

In this section we will describe a well known result that is usually referred to as *Small Object Argument*. This result consists of a transfinite construction to build factorisations of morphisms starting from a *set* of maps under some (co)completeness and smallness conditions. This result has been introduced by Quillen in [Qui67], then extended by Bousfield in [Bou77] and then refined by Garner in [Gar09]. We also refer to the reader the work of Hirschhorn in [Hir03], which presents the subject thoroughly.

We start this section with the following proposition, which is actually a corollary of Proposition 2.2.

Proposition 2.14. *Let $\mathcal{H}$ be a class of morphism in $\mathscr{C}$. Then $\left({}^{\boxslash}\mathcal{H}, ({}^{\boxslash}\mathcal{H})^{\boxslash}\right)$ and $\left({}^{\boxslash}(\mathcal{H}^{\boxslash}), \mathcal{H}^{\boxslash}\right)$ are weak prefactorisation systems.*

Although this proposition enables us to build weak prefactorisations systems from any class of morphisms, one must keep in mind that this construction might lead to trivial prefactorisation systems such as $(\mathsf{Iso}, \mathsf{All})$ and $(\mathsf{All}, \mathsf{Iso})$. One further improvement is finding conditions under which Proposition 2.14 allows us to build actual WFSs. In order to do that one needs a construction for factorisations and this is the aim of the Small Object Argument.

We recall the definition of transfinite composition.

Definition 2.15. *Let $\mathscr{C}$ be a category. By **transfinite composition** of a diagram*

$$\left(e^0_\alpha : X_0 \longrightarrow X_\alpha\right)_{\alpha<\lambda}$$

we mean the morphism e^0_λ of the colimit cocone $\left(X_\lambda, e^\alpha_\lambda : X_\alpha \longrightarrow X_\lambda\right)_{\alpha<\lambda}$ of such diagram, whenever such colimit exists.

Then we have the following result.

Proposition 2.16. *Let $\mathscr{C}$ be a category that has transfinite compositions and $\mathcal{H}$ a class of morphisms in $\mathscr{C}$. Then ${}^{\boxslash}\mathcal{H}$ is closed under transfinite compositions.*

Proof. We consider the following diagram

$$\left(e^0_\alpha : X_0 \longrightarrow X_\alpha\right)_{\alpha<\lambda}$$

where $e^0_\alpha \in {}^{\boxslash}\mathcal{H}$ for each $\alpha < \lambda$. We consider the transfinite composition of such sequence e^0_λ and a commutative square $[u,v] : e^0_\lambda \longrightarrow f$, with $f \in \mathcal{H}$. We can write the following diagram

$$\begin{tikzcd}
X_0 \arrow[rr, "u"] \arrow[dd, "e^0_\lambda"'] \arrow[dr, dashed, "e^0_\alpha"] & & A \arrow[dd, "f"] \\
& X_\alpha \arrow[ur, dashed, "d_\alpha"] \arrow[dl, "e^\alpha_\lambda"] & \\
X_\lambda \arrow[rr, "v"'] \arrow[uurr, dotted, bend right, "d"'] & & B.
\end{tikzcd}$$

The morphisms $(d_\alpha)_{\alpha<\lambda}$ are the diagonal liftings of e^0_α against f that arise since every morphism e^0_α in the diagram belongs to ${}^{\boxslash}\mathcal{H}$ and they satisfy the following identities $u = d_\alpha \cdot e^0_\alpha$. In particular $(d_\alpha)_{\alpha<\lambda}$ constitutes a cocone for the diagram $\left(e^0_\alpha\right)_{\alpha<\lambda}$, hence there exists the universal morphism $d : X_\lambda \longrightarrow A$ such that $d \cdot e^\alpha_\lambda = d_\alpha$. In particular we have that $d \cdot e^0_\lambda = u$ and

$$v \cdot e^\alpha_\lambda = f \cdot d_\alpha = f \cdot d \cdot e^\alpha_\lambda,$$

which yields that $v = f \cdot d$, since $\left(e^\alpha_\lambda\right)_{\alpha<\lambda}$ are jointly epic. We conclude that d is the diagonal lifting sought and that $e^0_\lambda \in {}^{\boxslash}\mathcal{H}$. $\square$

Definition 2.17. *Let λ be an ordinal. An $\mathcal{H}$-**cell** λ-**complex** is a diagram of the form*

$$\left(e^\alpha_\beta : E_\alpha \longrightarrow E_\beta\right)_{\alpha \le \beta < \lambda}$$

*such that $e^0_\beta = e^\alpha_\beta \cdot e^0_\alpha$ for any $\alpha \le \beta < \lambda$ and each e^0_α is a transfinite compositions of morphisms obtained from elements in $\mathcal{H}$ via pushouts, colimits and coproducts (in the dual sense of Definition 2.6). We will refer to an $\mathcal{H}$-cell λ-complex $\left(e^\alpha_\beta : E_\alpha \longrightarrow E_\beta\right)_{\alpha \le \beta \le \lambda}$ that extends to λ as $\mathcal{H}$-**cell** $\overline{\lambda}$-**complex**.*

Definition 2.18. *Let $\mathscr{C}$ be a category that admits transfinite compositions of $\mathcal{H}$-cell λ-complexes. An object W is **small relative to** $\mathcal{H}$ if there exists an ordinal κ such that for every $\lambda \ge \kappa$ and for every $\mathcal{H}$-cell λ-complex*

$$\left(e^\alpha_\beta : E_\alpha \longrightarrow E_\beta\right)_{\alpha \le \beta < \lambda}$$

and any morphism $u : W \longrightarrow E_\lambda$ there exists an ordinal $\gamma < \lambda$ and a morphism $u' \in \mathscr{C}\left(W, E_\gamma\right)$ such that $u = e^{\gamma+1}_\lambda \cdot e^\gamma_{\gamma+1} \cdot u'$.

Theorem 2.19 (Small Object Argument)**.** *Let $\mathscr{C}$ be a cocomplete category. Let $\mathcal{H}$ be a set of morphisms in $\mathscr{C}$ such that every domain of a morphism in $\mathcal{H}$ is small relative to $\mathcal{H}$. Then every morphism f in $\mathscr{C}$ factors as $f = p \cdot e$, where p is in $\mathcal{H}^{\boxslash}$ and e is in ${}^{\boxslash}(\mathcal{H}^{\boxslash})$.*

Proof. Since $\mathcal{H}$ is a set, there exists an ordinal λ such that every domain of a morphism in $\mathcal{H}$ is small relative to $\mathcal{H}$ with respect to some $\kappa \le \lambda$.

Consider $f : A \longrightarrow B$. We construct an $\mathcal{H}$-cell λ-complex $\left(e^{\alpha}_{\beta} : E_{\alpha} \longrightarrow E_{\beta}\right)_{\alpha \le \beta < \lambda}$ and a cocone $(p^{\alpha} : E_{\alpha} \longrightarrow B)_{\alpha < \lambda}$ for the diagram $\left(e^{0}_{\alpha}\right)_{\alpha < \lambda}$, i.e. such that

$$\begin{tikzcd}
E_0 \arrow[r, "e^0_\alpha"] \arrow[dr, "p^0"'] & E_\alpha \arrow[d, "p^\alpha"] \\
& B
\end{tikzcd}$$

for any $\alpha < \lambda$.

We start by setting $p^0 = f$. Let us consider an ordinal γ. We build the $\gamma + 1$ step as follows. We consider the commutative squares of the form

$$\begin{tikzcd}
A_j \arrow[r, "u_j"] \arrow[d, "h_j"'] & E_\gamma \arrow[d, "p^\gamma"] \\
B_j \arrow[r, "v_j"'] & B
\end{tikzcd} \tag{2.iii}$$

such that $h_j \in \mathcal{H}$ and we index these squares by $j \in I$. We consider the coproduct $\hat{h} = \coprod_{i \in I} h_i : \coprod_{i \in I} A_i \longrightarrow \coprod_{i \in I} B_i$, defined by the dual of Definition 2.6. Then we can build the following pushout diagram

$$\begin{tikzcd}
\coprod_{i\in I} A_i \arrow[r, "{[u_i]_{i\in I}}"] \arrow[d, "\hat{h}"'] & E_\gamma \arrow[d, "e^\gamma_{\gamma+1}"] \arrow[ddr, bend left, "p^\gamma"] & \\
\coprod_{i\in I} B_i \arrow[r, "q_\gamma"] \arrow[drr, bend right, "{[v_i]_{i\in I}}"'] & E_{\gamma+1} \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dotted, "p^{\gamma+1}"] & \\
& & B.
\end{tikzcd} \tag{2.iv}$$

The morphisms $[u_i]_{i \in I}$ and $[v_i]_{i \in I}$ are the universal morphisms induced by the two coproducts in $\mathscr{C}$. The morphism $e^{\gamma}_{\gamma+1} : E_{\gamma} \longrightarrow E_{\gamma+1}$ is a pushout of a coproduct of elements of $\mathcal{H}$, hence $e^{\gamma}_{\gamma+1}$ is in ${}^{\boxslash}(\mathcal{H}^{\boxslash})$, since $\mathcal{H} \subseteq {}^{\boxslash}(\mathcal{H}^{\boxslash})$ by Corollary 2.3 and weak orthogonal complements are closed under coproducts and pushout by Proposition 2.8. We define $e^{\alpha}_{\gamma+1} = e^{\gamma}_{\gamma+1} \cdot e^{\alpha}_{\gamma}$ for any $\alpha \le \gamma$. This yields that $e^{0}_{\gamma+1} = e^{\gamma}_{\gamma+1} \cdot e^{0}_{\gamma}$ belongs to ${}^{\boxslash}(\mathcal{H}^{\boxslash})$, since it is closed under composition. We remark that

$$\begin{cases} p^0 = p^{\gamma} \cdot e^0_{\gamma} = p^{\gamma+1} \cdot e^{\gamma}_{\gamma+1} \cdot e^0_{\gamma} = p^{\gamma+1} \cdot e^0_{\gamma+1}, \\ e^0_{\gamma+1} = e^{\gamma}_{\gamma+1} \cdot e^0_{\gamma} = e^{\gamma}_{\gamma+1} \cdot e^{\alpha}_{\gamma} \cdot e^0_{\alpha} = e^{\alpha}_{\gamma+1} \cdot e^0_{\alpha} \end{cases} \tag{2.v}$$

by inductive hypothesis.

Hence $\left(e^{\alpha}_{\beta} : E_{\alpha} \longrightarrow E_{\beta}\right)_{\alpha \le \beta < \gamma+1}$ is an $\mathcal{H}$-cell $(\gamma + 1)$-complex and $(p^{\alpha} : E_{\alpha} \longrightarrow B)_{\alpha < \gamma+1}$ is a cocone for the diagram $\left(e^0_{\alpha}\right)_{\alpha < \gamma+1}$.

Now for each limit ordinal $\kappa \le \lambda$ we have a $\mathcal{H}$-cell κ-complex $\left(e^{\alpha}_{\beta} : E_{\alpha} \longrightarrow E_{\beta}\right)_{\alpha\le\beta<\kappa}$ and a cocone $(p^{\alpha} : E_{\alpha} \longrightarrow B)_{\alpha<\kappa}$ such that $p^0 = p^{\alpha} \cdot e^0_{\alpha}$ by inductive hypothesis. Hence we may write the following diagram

$$\begin{tikzcd}
E_0 \arrow[rr, "e^0_{\alpha}"] \arrow[dr, "e^0_{\kappa}"'] \arrow[ddr, bend right, "p^0"'] & & E_{\alpha} \arrow[dl, "e^{\alpha}_{\kappa}"] \arrow[ddl, bend left, "p^{\alpha}"] \\
& E_{\kappa} \arrow[d, "p^{\kappa}"] & \\
& B, &
\end{tikzcd} \tag{2.vi}$$

where $(e^{\alpha}_{\kappa})_{\alpha\le\kappa}$ is a colimit cocone for the diagram $\left(e^0_{\alpha}\right)_{\alpha<\kappa}$ and p_{κ} is the universal morphism for the cocone $(p^{\alpha})_{\alpha<\kappa}$. In particular we have for any $\alpha \le \kappa$

$$\begin{cases} p^{\alpha} = p^{\kappa} \cdot e^{\alpha}_{\kappa} \\ e^0_{\kappa} = e^{\alpha}_{\kappa} \cdot e^0_{\alpha}. \end{cases}$$

Moreover, the morphism $e^0_{\kappa} : E_0 \longrightarrow E_{\kappa}$ is the transfinite composition of morphisms in $^{\boxslash}(\mathcal{H}^{\boxslash})$ and therefore it belongs to $^{\boxslash}(\mathcal{H}^{\boxslash})$ by Proposition 2.16. Thus we have that $\left(e^{\alpha}_{\beta} : E_{\alpha} \longrightarrow E_{\beta}\right)_{\alpha\le\beta\le\kappa}$ is an $\mathcal{H}$-cell $\overline{\kappa}$-complex and $(p^{\alpha} : E_{\alpha} \longrightarrow B)_{\alpha\le\kappa}$ is a cocone for the diagram $\left(e^0_{\alpha}\right)_{\alpha\le\kappa}$.

Then we have the factorisation

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e^0_{\lambda}"'] & & B \\
& E_{\lambda} \arrow[ur, "p^{\lambda}"'] &
\end{tikzcd}$$

with e^0_{λ} belonging to $^{\boxslash}(\mathcal{H}^{\boxslash})$. We prove that $p^{\lambda} : E_{\lambda} \longrightarrow B$ is in $\mathcal{H}^{\boxslash}$. We consider a square $[u,v] : h \longrightarrow p^{\lambda}$ with $h : H \longrightarrow H'$ in $\mathcal{H}$. By our smallness hypothesis, we know that there exists a morphism $u' \in \mathscr{C}\left(H, E_{\gamma}\right)$ such that $u = e^{\gamma+1}_{\lambda} \cdot e^{\gamma}_{\gamma+1} \cdot u'$, for some $\gamma < \lambda$. Thus we can write

$$\begin{tikzcd}
H \arrow[r, "u'"] \arrow[d, "h"'] \arrow[rrr, bend left, "u"] & E_{\gamma} \arrow[r, "e^{\gamma}_{\gamma+1}"] \arrow[d, "p^{\gamma}"] & E_{\gamma+1} \arrow[r, "e^{\gamma+1}_{\lambda}"] \arrow[dl, "p^{\gamma+1}"] & E_f \arrow[dll, "p^{\lambda}"] \\
H' \arrow[r, "v"'] \arrow[urr, dotted] & B. & &
\end{tikzcd} \tag{2.vii}$$

We remark that $[u',v] : h \longrightarrow p^{\gamma}$ is commutative, since $p^{\gamma+1} = p^{\lambda} \cdot e^{\gamma+1}_{\lambda}$ by (2.vi) and that $p^{\gamma} = p^{\gamma+1} \cdot e^{\gamma}_{\gamma+1}$ by construction in (2.iv). Therefore there exists $j \in I$ such that $[u',v] : h \longrightarrow p^{\gamma}$ is a square of the form described in diagram (2.iii) and it is part of the construction of $\hat{h}$ in the $\gamma+1$ step. This amounts to having that the following diagram is commutative

$$\begin{tikzcd}
H \arrow[r, "\sigma_h"] \arrow[d, "h"'] \arrow[rr, bend left, "u'"] & \coprod_{i\in I} A_i \arrow[r, "{[u_i]_{i\in I}}"] \arrow[d, "\hat{h}"] & E_\gamma \arrow[d, "e^\gamma_{\gamma+1}"'] \arrow[dr, "p^\gamma"] & \\
H' \arrow[r, "\sigma'_h"] \arrow[rrr, bend right=40, "v"'] & \coprod_{i\in I} B_i \arrow[r, "q_\gamma"] \arrow[rr, bend right, "{[v_i]_{i\in I}}"] & E_{\gamma+1} \arrow[r, "p^{\gamma+1}"] \arrow[ul, phantom, "\ulcorner", very near start] & B.
\end{tikzcd}$$

Hence $q_\gamma \cdot \sigma'_h$ is a possible dotted morphism in diagram (2.vii), therefore $e^{\gamma+1}_\lambda \cdot q_\gamma \cdot \sigma'_h$ is the diagonal lifting sought for the square $[u,v]$. We conclude that p^λ is in $\mathcal{H}^\boxslash$. $\square$

Corollary 2.20. *Let $\mathscr{C}$ be a cocomplete category. Let $\mathcal{H}$ be a set of morphisms in $\mathscr{C}$ such that every domain of a morphism in $\mathcal{H}$ is small relative to $\mathcal{H}$. Then $\left({}^\boxslash\left(\mathcal{H}^\boxslash\right), \mathcal{H}^\boxslash\right)$ is a* WFS.

Example 2.21. *We present now an application of this last constructive tool to build* WFS*s. Such example is well-known and presented for instance in* [Gar09, Rie14] *among others.*

Let us consider the category **Set** *and the* WFS $(\mathsf{Mono}, \mathsf{Epi})$ *mentioned in Example 2.11. Such* WFS *is generated through this process. Let us consider the set of maps $\mathcal{O} = \{! : \emptyset \longrightarrow *\}$, and a morphism $f : A \longrightarrow B$ such that $! \boxslash f$. Then every $b \in B$ defines a morphism $v_b : * \longmapsto b$, which makes the following diagram commute*

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d, "!"'] & A \arrow[d, "f"] \\
* \arrow[r, "v_b"'] \arrow[ur, dotted, "d"] & B,
\end{tikzcd}$$

where d exists by weak orthogonality. Then, $d(*) \in f^{-1}(B)$, which yields that f is epimorphic. Hence by uniqueness of weak orthogonal complements the WFS *generated is* $(\mathsf{Mono}, \mathsf{Epi})$. *Therefore, the small object construction induces the coproduct factorisation in Example 2.11.*

2.3 Functorial Factorisation Systems

Even if WFSs have proven to be quite rich and nuanced structures, one can wonder how to define a more complex notion of factorisation systems and whether it may induce some categorical way (read here functorial) to choose factorisations and diagonal liftings. The most general answer to such a question is provided by *functorial factorisation systems*. Such factorisation systems will in fact provide the components of factorisations functorially, which are still not unique. Since historically the notion of WFS drew interest later than the stronger and more established notion of orthogonal factorisation systems (for whose discussion we redirect the reader to the following Section 2.5), the first definition of functorial factorisation systems, which were modeled to grasp facets of the latter structure as for instance [KT93, JT99], were indeed stronger definitions than the one we present in this chapter. The definition that we will present in this section is introduced in [RT02], and is present in [Rie11] or [CLF16, CLF20].

We begin this section recalling some useful notation. Let $\mathscr{C}$ be any category. Then $\mathscr{C}^2 \times_{\mathscr{C}} \mathscr{C}^2$ is the category of pairs of composable morphisms, i.e. an object is a pair (f,g) such that $\mathrm{dom}\, g = \mathrm{cod}\, f$. Morphisms in $\mathscr{C}^2 \times_{\mathscr{C}} \mathscr{C}^2$ are triples of arrows $[u,v,w] : (f,g) \longrightarrow (f',g')$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "v"] \arrow[d, "g"'] & B' \arrow[d, "g'"] \\ C \arrow[r, "w"'] & C'. \end{tikzcd}$$

Then we have the following functors

$$\mathscr{C}^2 \times_{\mathscr{C}} \mathscr{C}^2 \begin{array}{c} \xrightarrow{\pi_1} \\ \xrightarrow{(-\cdot-)} \\ \xrightarrow{\pi_2} \end{array} \mathscr{C}^2 \begin{array}{c} \xrightarrow{\mathrm{dom}} \\ \xrightarrow{\mathrm{cod}} \end{array} \mathscr{C}.$$

The two morphisms π_1, π_2 are projections on the first and second components and $(-\cdot-)$ is the composition functor. Moreover, dom and cod are the functors that assign to each morphism its domain and its codomain, respectively. They are all trivially well-defined as functors.

Definition 2.22. *A* ***functorial factorisation system*** (FFS) *for a category $\mathscr{C}$ is a functor $F : \mathscr{C}^2 \longrightarrow \mathscr{C}^2 \times_{\mathscr{C}} \mathscr{C}^2$, which is a coretraction of $(-\cdot-)$, i.e. $(-\cdot-) \circ F = \mathrm{Id}_{\mathscr{C}^2}$.*

Any FFS F gives rise to the following components

$$L = \pi_1 \circ F \qquad R = \pi_2 \circ F \qquad K = \mathrm{cod} \circ L = \mathrm{dom} \circ R.$$

These functors assign to each morphism in $\mathscr{C}$ its left component, its right component and its middle object respectively. If we consider a commutative square $[u,v] : f \longrightarrow g$, then we have that the following diagram commutes by functoriality

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "Lf"'] & C \arrow[d, "Lg"] \\ Kf \arrow[r, "K(u{,}v)"] \arrow[d, "Rf"'] & Kg \arrow[d, "Rg"] \\ B \arrow[r, "v"'] & D. \end{tikzcd} \tag{2.viii}$$

Moreover, F gives rise to two transformations $\eta : \mathrm{Id}_{\mathscr{C}^2} \Longrightarrow R$ and $\varepsilon : L \Longrightarrow \mathrm{Id}_{\mathscr{C}^2}$. For any morphism $f : A \longrightarrow B$ these transformations are defined as follows

$$\begin{tikzcd}
A \arrow[r, "Lf"] \arrow[d, "f"'] \arrow[dr, phantom, "\eta_f"] & Kf \arrow[d, "Rf"] \\
B \arrow[r, "\mathrm{id}_B"'] & B,
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "Lf"'] \arrow[dr, phantom, "\varepsilon_f"] & A \arrow[d, "f"] \\
Kf \arrow[r, "Rf"'] & B.
\end{tikzcd}$$

Through the factorisation in (2.viii), for any commutative square $[u,v] : f \longrightarrow g$ one has the following commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "Lf"] \arrow[dr, "u"] \arrow[dd, "f"'] & & Kf \arrow[dr, "K(u{,}v)"] \arrow[dd, "Rf" near end] & \\
& C \arrow[rr, "Lg" near start] \arrow[dd, "g" near start] & & Kg \arrow[dd, "Rg"] \\
B \arrow[rr, "\mathrm{id}_B" near end] \arrow[dr, "v"'] & & B \arrow[dr, "v"] & \\
& D \arrow[rr, "\mathrm{id}_D"'] & & D.
\end{tikzcd} \tag{2.ix}$$

This diagram, considered from different perspectives, yields that η and ε are natural transformations.

A functorial factorisation system can be described as well by a triple (K,λ,ρ), where

- $K : \mathscr{C}^2 \longrightarrow \mathscr{C}$ is functor;
- $\lambda : \mathrm{dom} \Longrightarrow K$ and $\rho : K \Longrightarrow \mathrm{cod}$ are natural transformations such that for every $f \in \mathscr{C}^2$, $\rho_f \cdot \lambda_f = f$.

Another equivalent definition of a functorial factorisation system may be given by a pair of functors $L,R : \mathscr{C}^2 \longrightarrow \mathscr{C}^2$ such that

$$\mathrm{dom} \cdot L = \mathrm{dom} \qquad \mathrm{cod} \cdot R = \mathrm{cod} \qquad \mathrm{cod} \cdot L = \mathrm{dom} \cdot R$$

and for every $f \in \mathscr{C}^2$, $Rf \cdot Lf = f$.

In general there is no embedding between the class of functorial factorisation systems and the class of weak factorisation systems on a given category. We provide the following definition that links the two concepts.

Definition 2.23. *Let $(\mathcal{L},\mathcal{R})$ be a* WFS, *and (L,R) a* FFS *on a category $\mathscr{C}$. Then, if for every $f \in \mathscr{C}^2$, $Lf \in \mathcal{L}$ and $Rf \in \mathcal{R}$, we say that (L,R) is a **functorial realisation** of $(\mathcal{L},\mathcal{R})$. Moreover, any* WFS *that admits a functorial realisation is called a **functorial weak factorisation system**.*

We recall the following definition introduced in [Kel80] that we will use in the next result.

Definition 2.24. *A **pointed endofunctor** on a category $\mathscr{C}$ is a pair (T,η), with T an endofunctor of $\mathscr{C}$ and $\eta : \mathrm{Id}_{\mathscr{C}} \Longrightarrow T$ a natural transformation. A T-algebra for such a pointed endofunctor is a pair*

$(A, \alpha : TA \longrightarrow A)$ *such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & TA \arrow[d, "\alpha"] \\ & A. \end{tikzcd}$$

We will denote the class of all T-algebras by T-Alg to distinguish it from the class $\mathbb{T}$-Alg of algebras for a monad $\mathbb{T}$ in the sense of Eilenberg-Moore.

The dual notion of copointed endofunctor $(S, \varepsilon : S \Longrightarrow \mathrm{Id}_{\mathscr{C}})$, gives rise to a class of L–coalgebras, that we will denote by L-Coalg.

Theorem 2.25. *Let* $(\mathcal{L}, \mathcal{R})$ *be a* WFS *with functorial realisation* (L, R)*. Then* $\mathcal{L} = L$-Coalg *and* $\mathcal{R} = R$-Alg.

Proof. First, we notice that, if $f \in \mathcal{L}$ and $g \in \mathcal{R}$, we have the following diagonal morphisms for the squares η_f and ε_g

$$\begin{tikzcd} A \arrow[r, "Lf"] \arrow[d, "f"'] & Kf \arrow[d, "Rf"] \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "\rho_f"] & B, \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\mathrm{id}_C"] \arrow[d, "Lg"'] & C \arrow[d, "g"] \\ Kg \arrow[r, "Rg"'] \arrow[ur, "\lambda_g"] & D. \end{tikzcd} \tag{2.x}$$

These diagonal morphisms yield respectively

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "Lf"] & A \arrow[d, "f"] \\ B \arrow[r, "\rho_f"] \arrow[rr, bend right, "\mathrm{id}_B"'] & Kf \arrow[r, "Rf"] & B, \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "Lg"] \arrow[d, "g"'] \arrow[rr, bend left, "\mathrm{id}_C"] & Kg \arrow[r, "\lambda_g"] \arrow[d, "Rg"] & C \arrow[d, "g"] \\ D \arrow[r, equal] & D \arrow[r, equal] & D. \end{tikzcd}$$

Therefore $\left(f, \left[\mathrm{id}_A, \rho_f\right]\right)$ is an L-coalgebra for the copointed endofunctor (L, ε) and $(g, [\lambda_g, \mathrm{id}_D])$ is an R-algebra for the pointed endofunctor (R, η). Hence $\mathcal{L} \subseteq L$-Coalg and $\mathcal{R} \subseteq R$-Alg.

Now we consider an R-algebra $(g, [\alpha_0, \alpha_1])$. Then, considering the following diagram

$$\begin{tikzcd} C \arrow[r, "Lg"] \arrow[d, "g"'] \arrow[rr, bend left, "\mathrm{id}_C"] & Kg \arrow[r, "\alpha_0"] \arrow[d, "Rg"] & C \arrow[d, "g"] \\ D \arrow[r, equal] \arrow[rr, bend right, "\mathrm{id}_D"'] & D \arrow[r, "\alpha_1"] & D, \end{tikzcd} \tag{2.xi}$$

we have that α_1 is an identity. This yields that α_0 is a diagonal morphism for ε_g. Let $[u, v] : l \longrightarrow g$ a square with $l \in \mathcal{L}$. As seen in the first part of the proof there exists a diagonal morphism ρ_l for the square η_l. We can write the following diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "Ll"'] & C \arrow[d, "Lg"] \\
Kl \arrow[r, "K(u,v)"] \arrow[d, "Rl"] & Kg \arrow[d, "Rg"] \arrow[u, bend right, "\alpha_0"'] \\
Y \arrow[r, "v"'] \arrow[u, bend left, "\rho_l"] & D.
\end{tikzcd}$$

Then $\delta = \alpha_0 \cdot K(u,v) \cdot \rho_l$ is a diagonal morphism for $[u,v]$; in fact

$$\begin{cases} \delta \cdot l = \alpha_0 \cdot K(u,v) \cdot \rho_l \cdot l = \alpha_0 \cdot K(u,v) \cdot Ll = \alpha_0 \cdot Lg \cdot u = u \\ g \cdot \delta = g \cdot \alpha_0 \cdot K(u,v) \cdot \rho_l = Rg \cdot K(u,v) \cdot \rho_l = v \cdot Rl \cdot \rho_l = v. \end{cases}$$

Thus $l \boxslash g$ and therefore $g \in \mathcal{R}$. This proves that $\mathcal{R} = R$-Alg. Similarly one proves that L-Coalg $\subseteq \mathcal{L}$. This concludes the proof. $\square$

This theorem induces the following further result.

Theorem 2.26. *Let (L,R) be a* FFS. *If for every morphism f, $Lf \in L$-*Coalg *and $Rf \in R$-*Alg*, then $(L$-*Coalg*$, R$-*Alg*$)$ is a* WFS.

Proof. We consider $(L$-Coalg$, R$-Alg$)$ and $g \in R$-Alg. For every commutative square $[u,v] : f \longrightarrow g$, with $f \in L$-Coalg, we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "Lf"'] & C \arrow[d, "Lg"] \\
Kf \arrow[r, "K(u,v)"] \arrow[d, "Rf"] & Kg \arrow[d, "Rg"] \arrow[u, bend right, "\lambda_g"'] \\
B \arrow[r, "v"'] \arrow[u, bend left, "\rho_f"] & D.
\end{tikzcd}$$

We notice that $\delta = \lambda_g \cdot K(u,v) \cdot \rho_f$ is a diagonal morphism for the square. Hence R-Alg $\subseteq (L\text{-Coalg})^{\boxslash}$. On the other hand, if $g \boxslash L$-Coalg, then ε_g admits a diagonal lifting λ_g as in (2.x). This yields that $(g, [\rho_g, \mathrm{id}])$ is an R-algebra. Therefore R-Alg $\supseteq (L\text{-Coalg})^{\boxslash}$. This, together with the dual statement, yields that R-Alg $= (L\text{-Coalg})^{\boxslash}$ and L-Coalg $= {}^{\boxslash}(R\text{-Alg})$. $\square$

2.4 Algebraic Weak Factorisation Systems

In the previous section we have analysed the strengthening that functoriality brings to the table in the matter of factorisations. But along with the functorial description of factorisations, we also have that

FFS do not always induce an underlying WFS. Moreover, functoriality does not extend to the liftings ρ_{-} and λ_{-}, which play an important role in the previous results.

To reinforce the definition along these two directions algebraizations of FFS were introduced. The first introduced were *natural weak factorisation systems*, in [GT06]. Then the definition was refined by Garner in [Gar09] and later studied in depth in [BG16a, BG16b], who named them *algebraic weak factorisation systems*. We will focus in particular on this last formulation.

Definition 2.27. *An* ***algebraic weak factorisation system*** *(AWFS) is a FFS (L,R) such that*

- *(R,η) extends to a monad $\mathbb{R} = (R,\eta,\pi)$;*
- *(L,ε) extends to a comonad $\mathbb{L} = (L,\varepsilon,\sigma)$.*

Moreover, $\Delta = (\mathrm{cod}(\sigma),\mathrm{dom}(\pi)) : LR \longrightarrow RL$ is a natural transformation that constitutes a distributive law of the comonad over the monad. This distributivity amounts to the commutativity of the following diagram

$$\begin{tikzcd} LRR \arrow[r, "\Delta_R"] \arrow[d, "L\pi"'] & RLR \arrow[r, "R\Delta"] & RRL \arrow[d, "\pi_L"] \\ LR \arrow[rr, "\Delta"] \arrow[d, "\sigma_R"'] & & RL \arrow[d, "R\sigma"] \\ LLR \arrow[r, "L\Delta"] & LRL \arrow[r, "\Delta_L"] & RLL. \end{tikzcd} \tag{2.xii}$$

From here on we will denote an AWFS by $(\mathbb{L},\mathbb{R})$.

We remark that *natural weak factorisation systems*, as introduced by Grandis and Tholen, already entailed the idea of extending the (co)pointed endofunctor to (co)monads and the contribution of Garner was the addition of the distributivity law in the definition.

Remark 2.28. *Looking at the previous definition, we may write explicitly the natural transformation Δ as follows*

$$\begin{tikzcd} Kf \arrow[r, "\mathrm{cod}(\sigma_f)"] \arrow[d, "LRf"'] \arrow[dr, equal] & KLf \arrow[d, "RLf"] \\ KRf \arrow[r, "\mathrm{dom}(\pi_f)"'] & Kf. \end{tikzcd}$$

Commutativity of the two triangles is induced by the monad and the comonad axioms.

The first interesting property of AWFS is that by the monad and comonad axioms, for every morphism f, $Lf \in \mathbb{L}$-Coalg and $Rf \in \mathbb{R}$-Alg. Since $\mathbb{L}$-Coalg $\subseteq L$-Coalg and $\mathbb{R}$-Alg $\subseteq R$-Alg, then the hypotheses of Theorem 2.26 are satisfied and it induces that $(L\text{-Coalg}, R\text{-Alg})$ is a WFS.

We may regard the two subcategories $U : \mathbb{L}\text{-Coalg} \longrightarrow \mathscr{C}^2$ and $V : \mathbb{R}\text{-Alg} \longrightarrow \mathscr{C}^2$ as the left and the right classes associated with the AWFS $(\mathbb{L},\mathbb{R})$.

Analogously to what depicted in (2.xi), given any $\mathbb{R}$-algebra (f,α), then $\mathrm{cod}\alpha = \mathrm{id}_B$ and dually for any $\mathbb{L}$-coalgebra (g,β) we have that $\mathrm{dom}\beta = \mathrm{id}$. Thus $\mathbb{R}$-algebra and $\mathbb{L}$-coalgebra structures are uniquely determined by morphisms $r = \alpha_0 : Kf \longrightarrow \mathrm{dom}f$ and $s = \beta_1 : \mathrm{cod}g \longrightarrow Kg$ respectively. Moreover, given a commutative square $[h,k] : Uf \longrightarrow Vg$, we can write again the following diagram

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "LUf"'] & C \arrow[d, "LVg"] \\
Kf \arrow[r, "K(h,k)"] \arrow[d, "RUf"] & Kg \arrow[d, "RVg"] \arrow[u, bend right, "r"'] \\
B \arrow[r, "k"'] \arrow[u, bend left, "s"] & D.
\end{tikzcd} \tag{2.xiii}$$

and $\varphi_{f,g}(h,k) = r \cdot K(h,k) \cdot s$ is a diagonal lifting. Furthermore, such diagonal lifting is a canonical choice, in the sense that it is compatible with morphisms of the categories $\mathbb{L}$-Coalg and $\mathbb{R}$-Alg. In fact, given $\mu \in \mathbb{L}\text{-}\mathsf{Coalg}(f',f)$ and $\nu \in \mathbb{R}\text{-}\mathsf{Alg}(g,g')$, then considering the diagram

$$\begin{tikzcd}
A' \arrow[r, "\mathrm{dom}U\mu"] \arrow[d, "Uf'"'] & A \arrow[r, "h"] \arrow[d, "Uf"'] & C \arrow[r, "\mathrm{dom}V\nu"] \arrow[d, "Vb"] & C' \arrow[d, "Vg'"] \\
B' \arrow[r, "\mathrm{cod}U\mu"'] \arrow[urrr, dotted] & B \arrow[r, "k"'] \arrow[ur, dotted] & D \arrow[r, "\mathrm{cod}V\nu"'] & D'
\end{tikzcd}$$

the two dotted diagonal lifting, obtained as in (2.xiii), satisfy

$$\varphi_{f',g'}(\mathrm{dom}(V\nu) \cdot h \cdot \mathrm{dom}(U\mu), \mathrm{cod}(V\nu) \cdot k \cdot \mathrm{cod}(U\mu)) = \mathrm{dom}(V\nu) \cdot \varphi_{f,g}(h,k) \cdot \mathrm{cod}(U\mu).$$

Hence φ is a lifting operator, whose formal definition is the following.

Definition 2.29. *Let $U : \mathscr{A} \longrightarrow \mathscr{C}^2$ and $V : \mathscr{B} \longrightarrow \mathscr{C}^2$ be a pair of functors. Then a **lifting operator** of U against V is a natural transformation*

$$\varphi : \mathscr{C}^2(U\bullet, V\blacksquare) \Longrightarrow \mathscr{C}(\mathrm{cod}U\bullet, \mathrm{dom}V\blacksquare).$$

In particular the two classes $\mathbb{L}$-Coalg and $\mathbb{R}$-Alg are weakly orthogonal and each morphism of $\mathscr{C}$ admits an $(\mathbb{L}\text{-}\mathsf{Coalg}, \mathbb{R}\text{-}\mathsf{Alg})$-factorisation. Thus $(\mathbb{L}\text{-}\mathsf{Coalg}, \mathbb{R}\text{-}\mathsf{Alg})$ satisfies the first two axioms of Proposition 2.10. We can consider the retract closure of the two classes and we obtain that the pair $\left(\overline{\mathbb{L}\text{-}\mathsf{Coalg}}, \overline{\mathbb{R}\text{-}\mathsf{Alg}}\right)$ is the WFS underlying the AWFS $(\mathbb{L}, \mathbb{R})$.

We remark that the two retract-closed classes do not admit anymore a lifting operator in general.

2.5 Orthogonal Factorisation Systems

In this section we analyse and present the last and strongest definition of factorisation systems. *Orthogonal Factorisation Systems* were historically the first factorisation systems to appear and their introduction is due to [FK72]. Since then, the topic generated an entire branch of studies that created various sophisticated definitions and results and their theory entered with good reason the categorist's toolkit, as one can see in [AHS90]. Even if, at first glance, their definition differs very little from WFS, the bare addition of uniqueness makes them encompass and moreover surpass in strength all of the definitions that we have given so far. In this section we will present the orthogonality relation among maps, define orthogonal factorisation systems and describe some of their properties.

Definition 2.30. *A morphism e is* ***orthogonal*** *to a morphism m, denoted by* $e \perp m$*, if for every commutative square* $[u,v] : e \longrightarrow m$ *there exists a unique morphism* δ

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "e"'] & B \arrow[d, "m"] \\ C \arrow[r, "v"'] \arrow[ur, dashed, "\delta"] & D, \end{tikzcd}$$

such that the two triangles in the diagram commute. The morphism δ *is said to be the* ***unique diagonal lifting*** *of e against m for the square* $[u,v]$.

As we have discussed in 2.1, given a class of morphisms $\mathcal{H}$ in $\mathscr{C}$, one can build orthogonal complements $\mathcal{H}^\perp$ and ${}^\perp\mathcal{H}$. Uniqueness of diagonal liftings may be included in the proofs of Propositions 2.2 , 2.5 and Lemma 2.4, yielding the following results.

Proposition 2.31. *The pair* ${}^\perp(-)$ *and* $(-)^\perp$ *forms a Galois connection among the classes of morphisms in* $\mathscr{C}$ *partially ordered by the inclusion.*

Lemma 2.32. *Let f be a morphism in* $\mathscr{C}$*. Then the following conditions are equivalent:*

1. $f \perp g$ *for every morphism g in* $\mathscr{C}$*;*
2. $g \perp f$ *for every morphism g in* $\mathscr{C}$*;*
3. $f \perp f$*;*
4. *f is an isomorphism.*

Proposition 2.33. *Let* $\mathcal{H}$ *be a class of morphisms of* $\mathscr{C}$*. Then the following assertions hold*

1. $\mathcal{H}^\perp$ *and* ${}^\perp\mathcal{H}$ *contain all the isomorphisms of* $\mathscr{C}$*;*
2. $\mathcal{H}^\perp$ *is closed under composition;*
3. ${}^\perp\mathcal{H}$ *is closed under transfinite composition.*

Then an orthogonal factorisation system is defined as follows.

Definition 2.34. *An* ***orthogonal prefactorisation system*** *is a pair of classes of morphisms* $(\mathcal{E},\mathcal{M})$ *such that* $\mathcal{E}^\perp = \mathcal{M}$ *and* ${}^\perp\mathcal{M} = \mathcal{E}$*. If every morphism f admits an* $(\mathcal{E},\mathcal{M})$*-factorisation*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathcal{E}\ni e"'] & & B \\ & K_f, \arrow[ur, "m\in\mathcal{M}"'] & \end{tikzcd}$$

unique up to isomorphism, then $(\mathcal{E},\mathcal{M})$ *is called* ***orthogonal factorisation system*** *(OFS).*

We notice that as for WFSs the intersection of the two classes is given exactly by the isomorphisms of $\mathscr{C}$.

Again uniqueness of diagonal morphisms may be plugged in the proofs of Propositions 2.10 and 2.14, yielding the following results.

Proposition 2.35. *The pair $(\mathcal{E}, \mathcal{M})$ is an* OFS *if and only if the following conditions hold:*

1. *$\mathcal{E} \perp \mathcal{M}$, i.e. for every $e \in \mathcal{E}$ and $m \in \mathcal{M}$, then $e \perp m$;*

2. *every morphism f admits a unique $(\mathcal{E}, \mathcal{M})$-factorisation;*

3. *if* $\mathrm{id} = r \cdot s$, *then*

   (a) if $s \cdot f \in \mathcal{E}$, then $f \in \mathcal{E}$;

   (b) if $g \cdot r \in \mathcal{M}$, then $g \in \mathcal{M}$.

Proof. We just prove that if $(\mathcal{E}, \mathcal{M})$ is an OFS, then 3.(a) is true. The proof of 3.(b) is carried out in a similar fashion and the rest of the arguments can be easily adapted from the proof of Proposition 2.10. We consider $s \cdot f \in \mathcal{E}$ and r, s such that $\mathrm{id}_B = r \cdot s$. In fact, since existence of diagonal morphisms is already shown in Proposition 2.10, it remains to show that they are unique. Let d, d' be two diagonal liftings for a square $[u, v] : f \longrightarrow g$, with $g \in \mathcal{M}$. Then we can write

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[d, "f"'] & & C \arrow[dd, "g"] \\
B \arrow[urr, "d"] \arrow[urr, "d'"'] \arrow[d, "s"] & & \\
X \arrow[u, bend left, "r"] \arrow[r, "r"'] & B \arrow[r, "v"'] & D.
\end{tikzcd}$$

It induces that $d \cdot r$ and $d' \cdot r$ are two diagonal liftings of $s \cdot f$ against g, therefore $d \cdot r = d' \cdot r$ and, since r is a split epimorphism, $d = d'$. $\square$

Proposition 2.36. *Let $\mathcal{H}$ be a class of morphisms in $\mathscr{C}$. Then $\left({}^{\perp}\mathcal{H}, ({}^{\perp}\mathcal{H})^{\perp}\right)$ and $\left({}^{\perp}(\mathcal{H}^{\perp}), \mathcal{H}^{\perp}\right)$ are orthogonal prefactorisation systems.*

Proposition 2.37. *Let $(\mathcal{E}, \mathcal{M})$ be a* OFS. *Then it admits a functorial realisation which extends to an* AWFS.

Proof. We first prove that $(\mathcal{E}, \mathcal{M})$ admits a lifting operator. Given $e \in \mathcal{E}$ and $m \in \mathcal{M}$, for each $[u, v] : e \longrightarrow m$, there exists a unique choice for the map

$$[u, v] \longmapsto \varphi_{e,m}(u, v).$$

Furthermore, we consider the diagram

$$\begin{tikzcd}
A' \arrow[r, "a"] \arrow[d, "e'"'] & A \arrow[r, "u"] \arrow[d, "e"'] & C \arrow[r, "b"] \arrow[d, "m"] & C' \arrow[d, "m'"] \\
B' \arrow[r, "a'"'] \arrow[rrru, dotted] & B \arrow[r, "v"'] \arrow[ru, dotted] & D \arrow[r, "b'"'] & D'
\end{tikzcd}$$

where $e' \in \mathcal{E}$ and $m' \in \mathcal{M}$. Then

$$\varphi_{e',m'}\left(b \cdot u \cdot a, b' \cdot v \cdot a'\right) = b \cdot \varphi_{e,m}\left(u,v\right) \cdot a'$$

by uniqueness of diagonal morphisms lifting e' against m'. Now let $[u,v] : f \longrightarrow g$ and $[u',v'] : g \longrightarrow h$. Then we can write the diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "e_f"'] & C \arrow[r, "u'"] \arrow[d, "e_g"] & E \arrow[d, "e_h"] \\
K_f \arrow[r, "k"'] \arrow[rr, bend left, "k''"] \arrow[d, "m_f"'] & K_g \arrow[r, "k'"'] \arrow[d, "m_g"] & K_h \arrow[d, "m_h"] \\
B \arrow[r, "v"'] & D \arrow[r, "v'"'] & F
\end{tikzcd}$$

then $k'' = k' \cdot k$ by uniqueness of diagonal morphisms lifting e_f against m_h. This allows us to conclude that $(\mathcal{E}, \mathcal{M})$ has a functorial realisation that we denote (E, M). We aim to prove that (M, η) extends to a monad $\mathbb{M}$. We are seeking a natural transformation $\mu : MM \Longrightarrow M$. For any morphism f, $\mu_f : MMf \longrightarrow Mf$ is a commutative square, which is trivially an identity on codomains. We consider the following commutative diagram

$$\begin{tikzcd}
A \arrow[r, "Ef"] \arrow[d, "Ef \cdot EMf"'] & K_f \arrow[d, "Mf"] \arrow[dl, dotted, bend right, "EMf"'] \\
K_{Mf} \arrow[r, "MMf"'] \arrow[ur, dotted, bend right, "\mu_f^1"'] & B
\end{tikzcd}$$

where the two diagonal morphisms both exist since opposite sides of the diagram are orthogonal. Considering the following diagrams

$$
\begin{tikzcd}
A \arrow[rr, "Ef\cdot EMf"] \arrow[dd, "Ef\cdot EMf"'] \arrow[dr, "Ef"] & & K_{Mf} \arrow[dd, "MMf"] \\
& K_f \arrow[ur, "EMf"] \arrow[dr, "Mf"] & \\
K_{Mf} \arrow[rr, "MMf"'] \arrow[ur, "\mu^1_f"] & & B,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "Ef"] \arrow[dd, "Ef"'] \arrow[dr, "Ef\cdot EMf"] & & K_f \arrow[dd, "Mf"] \\
& K_{Mf} \arrow[ur, "\mu^1_f"] \arrow[dr, "MMf"] & \\
K_f \arrow[rr, "Mf"'] \arrow[ur, "EMf"] & & B,
\end{tikzcd}
$$

we get that $\mu^1_f \cdot EMf = \mathrm{id}_{K_f}$ and $EMf \cdot \mu^1_f = \mathrm{id}_{K_{Mf}}$ by uniqueness of diagonal liftings. Hence the natural transformation $\mu : MM \Longrightarrow M$ is a natural isomorphism, what we have proved yields trivially that the unit monad axioms hold for $\mathbb{M} = (M, \eta, \mu)$, the associativity axiom is trivially proved by a direct calculation. Likewise one can prove that $\mathbb{E} = (E, \varepsilon, \sigma)$ is a comonad and $\sigma : E \Longrightarrow EE$ is a natural isomorphism. Hence, since μ and σ are natural isomorphisms, then the distributivity law axiom in (2.xii) is satisfied. □

We have proved that OFSs on a given category are a subclass of its AWFSs. Moreover, we can characterise OFSs among AWFSs by the structure of the (co)monads that constitute them. Before stating this characterisation, which is due to [GT06], we recall the definition of *idempotent monads*.

Definition 2.38. *A monad $\mathbb{T} = (T, \eta, \mu)$ is* ***idempotent****, if one of the following equivalent conditions holds:*

1. $T\eta = \eta_T$*;*
2. $\mu : TT \Longrightarrow T$ *is a natural isomorphism;*
3. *all components of* μ *are monomorphic;*
4. $\mathbb{T}\text{-}\mathsf{Alg} \hookrightarrow \mathscr{C}^2$ *is full and faithful.*

Theorem 2.39. *Let $(\mathbb{E}, \mathbb{M})$ be an* AWFS *on a category $\mathscr{C}$. Then the following conditions are equivalent:*

1. *the underlying* WFS $(\mathcal{E}, \mathcal{M})$ *is an* OFS*;*
2. *liftings of* $\mathbb{E}$*-coalgebras against* $\mathbb{M}$*-algebras are unique;*
3. $\mathbb{M}$ *is an idempotent monad;*
4. $\mathbb{E}$ *is an idempotent comonad.*

Proof. 1. $\Rightarrow$ 2. This is trivially true since $\mathbb{E}\text{-}\mathsf{Coalg} \subseteq \mathcal{E}$ and $\mathbb{M}\text{-}\mathsf{Alg} \subseteq \mathcal{M}$.

2. ⇒ 3. We consider an $\mathbb{E}$-coalgebra f. Then we can write for η_f the following factorisation

$$\begin{tikzcd} A \arrow[r, "Ef"] \arrow[d, "f"'] \arrow[dr, phantom, "\eta_f"] & K_f \arrow[d, "Mf"] \\ B \arrow[r, "\mathrm{id}_B"'] & B \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd} A \arrow[r, "Ef"] \arrow[d, "Ef"'] & K_f \arrow[d, "EMf"] \\ K_f \arrow[r, shift left, "EMf"] \arrow[r, shift right, "K(Ef{,}\mathrm{id}_B)"'] \arrow[d, "Mf"'] & K_{Mf} \arrow[d, "MMf"] \\ B \arrow[r, "\mathrm{id}_B"'] & B \end{tikzcd}$$

Uniqueness of liftings of Ef against Mf, which is an $\mathbb{M}$-algebra, induces that $M(Ef,\mathrm{id}_B) = EMf$. Hence

$$M\eta_f = (K(Ef,\mathrm{id}_B),\mathrm{id}_B) = (EMf,\mathrm{id}_B) = \eta_{Mf}.$$

Therefore $\mathbb{M}$ is idempotent.

2. ⇒ 4. This implication is proved similarly to the previous one.

4. ⇒ 2. Idempotency of $\mathbb{E}$ yields in particular that $V : \mathbb{E}\text{-}\mathsf{Coalg} \longrightarrow \mathscr{C}^2$ is fully faithful, thus, for every $(f : A \to B, \beta) \in \mathbb{E}\text{-}\mathsf{Coalg}$, the square $[f,\mathrm{id}_B] : f \longrightarrow \mathrm{id}_B$ is an $\mathbb{E}$-coalgebra morphism. Let $[u,v] : f \longrightarrow g$ a commutative square with g coming from an $\mathbb{M}$-algebra, then, for any diagonal morphism j for $[u,v]$, we have

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd} \qquad = \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[r, "j"] \arrow[d, "\mathrm{id}_B"'] & C \arrow[d, "g"] \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, dotted, "\varphi_{f,g}(u{,}v)"] & B \arrow[r, "v"'] \arrow[ur, dotted, "\varphi_{\mathrm{id}_B,g}(j{,}v)"'] & D \end{tikzcd}$$

and by naturality, we have that $j = \varphi_{\mathrm{id}_B,g}(j,v) = \varphi_{f,g}(u,v)$.

3. ⇒ 2. The proof is analogous to the previous.

2. ⇒ 1. By the previous arguments, we have that our hypothesis 2. yields that $\mathbb{E}$ and $\mathbb{M}$ are both idempotent, which means that $\mathbb{E}\text{-}\mathsf{Coalg} \hookrightarrow \mathscr{C}^2$ and $\mathbb{M}\text{-}\mathsf{Alg} \hookrightarrow \mathscr{C}^2$ are fully faithful. Therefore retracts of $\mathbb{E}$-coalgebras are $\mathbb{E}$-coalgebras and retracts of $\mathbb{M}$-algebras are $\mathbb{M}$-algebras. Then condition 2. yields that $\overline{\mathbb{E}\text{-}\mathsf{Coalg}} \perp \overline{\mathbb{M}\text{-}\mathsf{Alg}}$, by Proposition 2.35, whose conditions are satisfied, we conclude that $(\mathbb{E}\text{-}\mathsf{Coalg}, \mathbb{M}\text{-}\mathsf{Alg})$ is an OFS. □

Example 2.40. *We remark that, besides the trivial examples* $(\mathsf{All},\mathsf{Iso})$ *and* $(\mathsf{Iso},\mathsf{All})$*, also* $(\mathsf{Epi},\mathsf{Mono})$ *in Example 2.12 is an* OFS*. In fact, it is easy to notice that any pair of diagonal morphisms in*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "\mathsf{Epi}\ni e"'] & C \arrow[d, "m\in\mathsf{Mono}"] \\ B \arrow[r, "v"'] \arrow[ur, shift left, "d"] \arrow[ur, shift right, "d'"'] & D \end{tikzcd}$$

must actually coincide. Such OFS *yields for any morphism the factorisation*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\overline{f}"'] & & B \\ & \mathrm{Im} f \arrow[ur, "i_{\mathrm{Im} f}"'] & \end{tikzcd}$$

which is unique up to isomorphism. This is a particular case of the most general known fact that in any regular category the pair $(\mathsf{RegEpi}, \mathsf{Mono})$ *is an* OFS.

2.6 Lax Orthogonal Factorisation Systems

In this section we consider a more recent structure introduced in [CLF16, CLF20]. The reason to take some time to mention this work is twofold. The first reason is that this study, which is close to the author's working environment, has been the first and main inspiration that ignited the study and research on the various topics contained in this work, original and known. The second reason is that this work provides another significant interplay between higher structures, such as **Ord**-enrichments or 2-categories, and factorisation systems, which is the perk of this present investigation as well. Our goal in this section is to give an overview on the definition of a *lax orthogonal factorisation system* and briefly describe the example that inspired their introduction.

We begin by recalling some definitions on 2-categories and enriched categories.

Definition 2.41. *A* ***2-category*** *is given by the following elements*

- *a class of objects, also called 0-cells;*
- *a class of morphisms between objects, also called 1-cells;*
- *a class of 2-morphisms, or 2-cells, between 1-cells;*

and such that 0-cells and 1-cells constitute a category and 2-cells have two associative and unital composition laws

- *a vertical composition that operates as follows*

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "f"{name=F}] \arrow[r, "g"{description, name=G}] \arrow[r, bend right=60, "h"'{name=H}] & \bullet \arrow[Rightarrow, from=F, to=G, phantom, "\Downarrow\alpha"] \arrow[Rightarrow, from=G, to=H, phantom, "\Downarrow\beta"] \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \bullet \end{tikzcd}$$

- *a horizontal composition that operates as follows*

$$\begin{tikzcd} \bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} \bullet \arrow[r, bend left, "f'\cdot f"] \arrow[r, bend right, "g'\cdot g"'] \arrow[r, phantom, "\Downarrow\beta*\alpha"] & \bullet \end{tikzcd}$$

Moreover, horizontal composition must preserve unit 2-cells and satisfies, together with the vertical composition, the following interchange law

$$(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha)$$

for any diagram

$$\begin{tikzcd} \bullet \arrow[r, bend left=50, "f"] \arrow[r, "g" description] \arrow[r, bend right=50, "h"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2ex, "\Downarrow\beta"] & \bullet \arrow[r, bend left=50, "f'"] \arrow[r, "g'" description] \arrow[r, bend right=50, "h'"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\gamma"] \arrow[r, phantom, shift right=2ex, "\Downarrow\delta"] & \bullet. \end{tikzcd}$$

Definition 2.42. *A* ***2-functor*** *is a map between two 2-categories $F : \mathscr{C} \longrightarrow \mathscr{D}$ such that F is a functor on 0-cells and 1-cells and it is also compatible with horizontal and vertical compositions of 2-cells.*

Definition 2.43. *A* ***2-natural transformation*** *$\alpha : F \Longrightarrow G$, between the 2-functors $F, G : \mathscr{C} \longrightarrow \mathscr{D}$, is given by a natural family of morphisms $(\alpha_X \in \mathscr{D}(FX, GX))_{X \in \mathsf{Ob}(\mathscr{C})}$ such that for each 1-cell $f : A \longrightarrow B$ in $\mathscr{C}$ there exists the identity 2-cell α_f as in the diagram*

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\alpha_f \Nearrow"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"'] & GB. \end{tikzcd}$$

Moreover, the association $f \longmapsto \alpha_f$ must preserve compositions and identity.

Then we can define a 2-monad as follows.

Definition 2.44. *A* ***2-monad*** *is given by a monad $\mathbb{T} = (T, \eta, \mu)$ on a 2-category $\mathscr{C}$ such that T is a 2-functor, η, μ are 2-natural transformations and the monad axioms are satisfied by the existence of identity 2-cells.*

We remark that the previous definitions are given in the sense of *strict* 2-categories, as they are called in opposition to *weak* 2-categories that entail only 2-isomorphisms whereas we evoked the existence of identity 2-cells in the definitions.

We can now provide the definition of lax idempotent monads for a 2-category, known also as Koch-Zöberlein monads or doctrines recalling their introduction in [Zöb76, Koc95].

Definition 2.45. *A 2-monad $\mathbb{T} = (T, \eta, \mu)$ is* ***lax idempotent*** *if any of the following conditions hold.*

- *$T\eta \dashv \mu$ with identity unit;*
- *$\mu \dashv \eta_T$ with identity counit;*
- *each $a : TA \longrightarrow A$ is a $\mathbb{T}$-algebra structure if and only if $a \dashv \eta_A$ with identity counit;*
- *the forgetful functor $U_{\text{lax}} : \mathbb{T}\text{-}\mathsf{Alg}_{\text{lax}} \longrightarrow \mathscr{C}$ is fully faithful;*

- *for any pair of* $\mathbb{T}$*-algebras* A, B*, every morphism* $f : UA \longrightarrow UB$ *admits a unique structure of a lax morphism of* $\mathbb{T}$*-algebras.*

The equivalence of such conditions may be found in [KL97]. Then we can provide the main definition introduced in [CLF16].

Definition 2.46. *A* ***lax orthogonal factorisation system*** *(*LOFS*) on a 2-category is an* AWFS $(\mathbb{L}, \mathbb{R})$ *such that* $\mathbb{L}$ *is a lax idempotent 2-comonad and* $\mathbb{R}$ *a lax idempotent 2-monad.*

The previous definitions may be presented in a sort of "*skeletal*" version. In fact, if a 2-category may be seen as a category enriched over the cartesian monoidal category Cat, we can consider categories enriched over the category of partial orders **Ord** (as we will from here on denote it). An **Ord**-enriched category, or **Ord**-category, is a category $\mathscr{C}$ such that for every $A, B \in \mathsf{Ob}(\mathscr{C})$, then $\mathscr{C}(A, B)$ is an object on **Ord**. In particular **Ord**-functors are monotone functors, i.e. $F : \mathscr{C} \longrightarrow \mathscr{D}$ a functor such that if $f \leq g$, then $Ff \leq Fg$.

Then we may rewrite Definition 2.45 for **Ord**-categories.

Definition 2.47. *A monad* $\mathbb{T} = (T, \eta, \mu)$ *on an* **Ord***-category* $\mathscr{C}$ *is* ***lax idempotent*** *if it satisfies any of the following equivalent conditions.*

- $T\eta \cdot \mu \leq \mathrm{Id}$;
- $\mathrm{Id} \leq \eta_T \cdot \mu$;
- $T\eta \leq \eta_T$;
- *for any* $\mathbb{T}$*-algebra* $a : TA \longrightarrow A$*, then* $\mathrm{id}_{TA} \leq \eta_A \cdot a$;
- *for any* $\mathbb{T}$*-algebras* (A, a) *and* (B, b)*, and any morphism* $f \in \mathscr{C}(A, B)$*, then* $b \cdot Tf \leq f \cdot a$.

We cite this application on the **Ord**-enriched context for two reasons. The first is the relevance with the setting of our study. The second is that the example that led to the introduction of LOFSs rose in the **Ord**-enriched context in [CCM12]. The subject of the paper is the category $\mathbf{Top}_0$ of **T0**-topological spaces and filter monads. Such spaces are considered equipped with the dual of the *specialisation order*, a partial order defined for any space by $x \leq y$ if and only if $y \in \overline{\{x\}}$. Then the study proves that filter monads and their slicings are lax idempotent and concludes that they determine a factorisation system.

The notable feature of LOFSs is that they are equipped with a particular type of lifting operators.

Definition 2.48. *Let* $U : \mathscr{A} \longrightarrow \mathscr{C}^2$ *and* $V : \mathscr{B} \longrightarrow \mathscr{C}^2$ *be locally monotone functors. A* ***KZ-lifting operator*** *is a lifting operator such that for any* $\varphi_{a,b}(h, k)$*, if there exists another diagonal lifting* d *for the square* $[h, k] : Ua \longrightarrow Vb$*, then* $\varphi_{a,b}(h, k) \leq d$.

Therefore KZ-liftitng operators provide a natural choice of diagonal morphisms which are minimal with respect to all other diagonal morphisms for any given square.

The importance of KZ-lifting operators is then illustrated by [CLF20, Theorem 8.3].

Theorem 2.49. *Given an* AWFS $(\mathbb{L}, \mathbb{R})$ *for an* **Ord***-category* $\mathscr{C}$*, the following are equivalent*

1. $(\mathbb{L},\mathbb{R})$ *is a* LOFS*;*

2. *the lifting operator from the forgetful functor* $U : \mathbb{L}\text{-}\mathsf{Coalg} \longrightarrow \mathscr{C}^2$ *to* $V : \mathbb{R}\text{-}\mathsf{Alg} \longrightarrow \mathscr{C}^2$ *is a KZ-lifting operator.*

In particular we can conclude that in the enriched context OFSs are lax orthogonal, by uniqueness of diagonal liftings.

2.7 Enriched factorisation systems

In conclusion of this chapter we would like to mention another type of factorisation systems that are interlinked with higher structures on a category. This factorisation systems were introduced by Riehl in [Rie14] as a generalisation of [Day74, LW14].

The setting here is given by a complete and cocomplete $\mathscr{V}$-enriched category $\mathscr{K}$, where $\mathscr{V}$ is a complete and cocomplete symmetric monoidal closed category. Riehl introduces the following definition.

Definition 2.50. *Let* $f : A \longrightarrow B$ *and* $g : C \longrightarrow D$ *be two morphisms of a* $\mathscr{V}$*-category* $\mathscr{K}$*. The two morphisms are said to satisfy the* $\mathscr{V}$***-enriched lifting property****, denoted* $f \boxslash g$*, if the canonical map* ω *admits a section* δ

$$\begin{tikzcd}
\mathscr{K}(B,C) \arrow[dr, "\omega"] \arrow[drr, bend left, "-\cdot f"] \arrow[ddr, bend right, "g\cdot -"'] & & \\
& \mathscr{C}^2(f,g) \arrow[r] \arrow[d] \arrow[ul, dashed, bend right, "\delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{K}(A,C) \arrow[d, "g\cdot -"] \\
& \mathscr{K}(B,D) \arrow[r, "-\cdot f"'] & \mathscr{K}(A,D).
\end{tikzcd} \tag{2.xiv}$$

This last definition is by all means an orthogonality-like property between morphisms. Again we can build complements similarly to what was presented before and recall the following definition.

Definition 2.51. *An* ***enriched weak factorisation system*** *is a pair of classes of morphisms* $(\mathcal{S},\mathcal{D})$ *such that* $\mathcal{S} = {}^{\boxslash}\mathcal{D}$*,* $\mathcal{D} = \mathcal{S}^{\boxslash}$ *and every morphism in* $\mathscr{K}$ *admits an* $(\mathcal{S},\mathcal{D})$*-factorisation.*

These definitions are therefore a generalisation of the following definition due to Day and Lucyshyn-Wright.

Definition 2.52. *Let* $f : A \longrightarrow B$ *and* $g : C \longrightarrow D$ *be two morphisms of a* $\mathscr{V}$*-category* $\mathscr{K}$*. The two morphisms are said to satisfy the* $\mathscr{V}$***-enriched unique lifting property****, denoted* $f \underline{\perp} g$*, if the canonical map* ω *in* (2.xiv) *is an isomorphism.*

An ***enriched orthogonal factorisation system*** *is a pair of classes of morphisms* $(\mathcal{S},\mathcal{D})$ *such that* $\mathcal{S} = {}^{\underline{\perp}}\mathcal{D}$*,* $\mathcal{D} = \mathcal{S}^{\underline{\perp}}$ *and every morphism in* $\mathscr{K}$ *admits an* $(\mathcal{S},\mathcal{D})$*-factorisation, which is unique up to isomorphism.*

Chapter 3

Theory of partial maps

For the purposes of our work we will be interested in particular in categories of partial maps considered as **Ord**-categories.

Along the chapter and later on we will also refer to discrete **Ord**-enrichments; by this we mean those categories where the partial order relation among morphisms coincides with the identity. In particular we observe that any category may be equipped with a discrete **Ord**-enrichment.

3.1 Interest in categories of partial maps

Partial maps are a very common object of study in mathematics and this is evident from the first steps in undergraduate mathematics. In fact, we come across partial maps in many branches of mathematics noticing that they provide a natural environment and a good behaved tool to develop mathematical knowledge.

This emerges when one considers calculus, where the use of partial morphisms is ubiquitous and has called many to a formalisation of partial maps as in [Men55]. However, partial maps are also quite present in the study of topology, a field that has made a great use of the categorical benchmark and this three-fold interest started to emerge already in [BB78].

This attention for partial morphisms is not limited to studies that deal with the most evidently geometrical facets of mathematics, but it spreads to more combinatorical investigations as shown by the quite prolific literature on partial functions and restrictions that was produced by semigroup theorists in the years, see for instance [SS67, Man06]. In particular, as remarked in [CM09], categorists and semigroup theorists have studied restriction semigroups, regrettably on parallel tracks for much of the earlier stages, unknowing of the shared interest on the algebraic and categorical axiomatisation of partiality.

Computer science is another field of investigation that has in time absorbed quite a good deal of categorical formalisation and this community looks with great interest at partiality. Indeed, this is witnessed by a remarkable number of publications on the subject [Plo85, Mog86, AL88, FP94, Fio04]. Computer science has fueled quite some discussion on a categorical approach to partial morphisms and domain theory. The interest of computer scientists in partial maps stems from some very structural topics of investigation, such as recursion theory. Partial maps provide again a suitable instrument to give a more abstract and general interpretation of programs and the problems of termination.

In this context it is worth to cite the work of Robinson and Rosolini [RR88], which provides a rich survey on the rising interest in categorical study of partial morphisms. We remark in particular their keen eye on both computer science community and the enriched categorical perspective.

Furthermore, we remark that an **Ord**-enrichment, as well as a 2-categorical approach, has always accompanied the development of categories of partial maps, as we can see in [AL88, Car87, Fio95]. This points out to the fact that partial orders, besides being naturally built-in in the concept of domain theory and partial morphisms, also provide a complementary and relevant perspective on the study and use of categories of partial maps.

We shall not fail to report that the interest in partiality and partial maps among the categorical community was not extinguished within these works, but has lived on and it is still active. To this purpose we cite the work of Cockett and Lack, who introduced *restriction categories* in [CL02, CL03]. Restriction categories constitute a more general formulation of categories endowed with partiality, which releases the definition from the use of pullbacks. Such structure substitute the use of spans and pullbacks with the introduction of a *restriction operator*, satisfying a suitable set of axioms.

We conclude remarking that in the current work we will present and use the classical formulation of categories of partial morphisms, in accordance with [Fio95, Fio04].

3.2 Definitions and properties

First we recall in this section the definitions and properties of categories of partial maps. Some of the features that we present may be found in [RR88, Fio95, Fio04].

The categorical definition of partial maps is motivated by the well established theory of partial functions, in particular between sets. A partial map between sets is generally given by a partial domain, where the function is defined, and the function itself. For this reason the first step is to define a good notion of subobjects. We consider a category $\mathscr{C}$, then a subobject of an object A is an equivalence class of monomorphisms $m : S \longrightarrow A$; two monomorphisms $m : S \longrightarrow A$ and $m' : S' \longrightarrow A$ are equivalent if there exists an isomorphism i such that $m = m' \cdot i$. In particular, to enable us to compose partial maps, we require that subobjects are closed under composition, that pullbacks along subobjects exist and that subobjects are preserved by pullbacks.

We consider $\mathscr{C}$ a category and $\mathcal{S}$ a good class of subobjects, i.e. closed under composition, pullback stable, and that contains all sections. We require that pullbacks along morphisms in $\mathcal{S}$ exist in $\mathscr{C}$. We will refer to the class $\mathcal{S}$ as the class of **admissible subobjects**. Then a **partial map** is an equivalence class of spans

$$\begin{tikzcd} D_f \arrow[d, tail, "\sigma_f"'] \arrow[dr, "\varphi_f"] & \\ A \arrow[r, "f"', "|" marking] & B \end{tikzcd}$$

with σ_f a subobject in $\mathcal{S}$ and φ_f a generic morphism in $\mathscr{C}$. Two spans (σ_f, φ_f) and (σ'_f, φ'_f) are equivalent if there exists an isomorphism i such that $\sigma'_f = \sigma_f \cdot i$ and $\varphi'_f = \varphi_f \cdot i$. We will often use from here on the notation $D_{\_}$, $\sigma_{\_}$ and $\varphi_{\_}$ to refer to the partial domain and the partial components of a partial map respectively. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two partial morphisms. Then their composition is

$$
\begin{tikzcd}
D_{g\cdot f} \arrow[d, tail, "\varphi_f^*(\sigma_g)"'] \arrow[dr, "\sigma_g^*(\varphi_f)"] & & \\
D_f \arrow[d, tail, "\sigma_f"'] \arrow[dr, "\varphi_f"] & D_g \arrow[d, tail, "\sigma_g"'] \arrow[dr, "\varphi_g"] & \\
A \arrow[r, "f"', "|" marking] & B \arrow[r, "g"', "|" marking] & C,
\end{tikzcd}
\tag{3.i}
$$

where the square is a pullback. Moreover, this composition law is trivially associative and for every object A in $\mathscr{C}$ there exists an identity represented by the span $(\mathrm{id}_A, \mathrm{id}_A)$. Hence we have that objects of $\mathscr{C}$ and the partial maps that we have just described form a category that we will denote by $\mathcal{P}_{\mathcal{S}}(\mathscr{C})$ or $\mathcal{P}(\mathscr{C})$, where it cannot generate any ambiguity.

Furthermore, given a category of partial maps $\mathcal{P}_{\mathcal{S}}(\mathscr{C})$ there exists an embedding

$$
\begin{aligned}
E : \mathscr{C} &\longrightarrow \mathcal{P}_{\mathcal{S}}(\mathscr{C}) \\
A &\longrightarrow A \\
f &\longrightarrow (\mathrm{id}_A, f)
\end{aligned}
$$

which is an identity on objects and faithful. The morphisms in the image of E, i.e. those partial morphism of the form (id_A, f), are called **total maps** and form the class Tot.

Proposition 3.1. *Let $\mathcal{P}(\mathscr{C})$ be a category of partial maps. For any pair of composable morphisms f and g, if $g \cdot f$ is total, then f is total.*

Proof. We consider diagram (3.i). Since $\mathcal{S}$ is stable under pullbacks, we have that $\varphi_f^*(\sigma_g)$ belongs to $\mathcal{S}$. Hence $D_{g\cdot f}$ is a subobject of D_f, therefore if $D_{g\cdot f} = A$ then $D_f = A$. □

Example 3.2. *The first example that we can consider is* **Set**. *In fact* **Set** *is a complete category, thus it has pullbacks, and monomorphisms are trivially stable under pullbacks. Hence, if we consider $\mathcal{S} = \mathsf{Mono}$, then $\mathcal{P}_{\mathsf{Mono}}(\mathbf{Set})$ is the category of sets and partial maps as usually intended.*

Every category of partial maps comes equipped with a partial order between morphisms induced by the partial order among subobjects. We write $f \leq g$ in $\mathcal{P}(\mathscr{C})$ if f is a domain restriction of g; namely if there exists an admissible subobject s making the following diagram commute:

$$
\begin{tikzcd}
D_f \arrow[d, tail, "s"] \arrow[dd, bend right=50, "\sigma_f"'] \arrow[ddr, "\varphi_f"] & \\
D_g \arrow[d, tail, "\sigma_g"'] \arrow[dr, "\varphi_g"] & \\
A \arrow[r, shift left, "f", "|" marking] \arrow[r, shift right, "g"', "|" marking] & B.
\end{tikzcd}
\tag{3.ii}
$$

Furthermore, if $\mathscr{C}$ is an **Ord**-category, then the partial order among its morphisms, that we denote by $\sqsubseteq$, gives rise to a partial order among partial maps. We will write $f \leq g$ if there exists an admissible subobject morphism s, which allows us to write the following diagram

$$
\begin{tikzcd}
D_f \arrow[d, tail, "s"] \arrow[dd, bend right=50, tail, "\sigma_f"'] \arrow[ddr, "\varphi_f"] & \\
D_g \arrow[d, tail, "\sigma_g"] \arrow[dr, "\varphi_g"] \arrow[r, phantom, "\sqsupseteq"] & \\
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B.
\end{tikzcd}
\tag{3.iii}
$$

It is clear that if we consider $\mathscr{C}$ with a discrete **Ord**-enrichment, namely all the 2-cells are equalities, then the two partial orders coincide.

Now we are interested in analysing whether these partial orders among partial maps constitute an **Ord**-enrichment. To do so we need to check that composition is monotone with respect to those partial order relations. This proves to be true for the discrete **Ord**-enrichment, while, in the general case, it requires some further assumptions on the admissible subobjects.

First we observe that in general the partial orders that we defined above respect compositions on the right without any further assumption. We prove it in the following proposition for the partial order $\leq$, and this applies in particular to $\leq$.

Proposition 3.3. *Let $\mathscr{C}$ be an* **Ord***-category and $\mathcal{P}(\mathscr{C})$ a category of partial maps. If $f : A \nrightarrow B$ and $g' \leq g : B \nrightarrow C$, then $g' \cdot f \leq g \cdot f$.*

Proof. We write explicitly the diagram that depicts the composition

$$
\begin{tikzcd}
D_{g'\cdot f} \arrow[d, tail, "s'"] \arrow[dr, "\sigma^*_{g'}(\varphi_f)"] \arrow[dd, bend right=60, tail, "\varphi^*_f(\sigma_{g'})"'] & & \\
D_{g\cdot f} \arrow[d, tail, "\varphi^*_f(\sigma_g)"'] \arrow[dr, "\sigma^*_g(\varphi_f)"] & D_{g'} \arrow[d, tail, "s"] \arrow[dd, bend right, tail, "\sigma_{g'}"'] \arrow[ddr, bend left, "\varphi_{g'}"] & \\
D_f \arrow[d, tail, "\sigma_f"] \arrow[dr, "\varphi_f"] & D_g \arrow[d, tail, "\sigma_g"] \arrow[dr, "\varphi_g"] \arrow[r, phantom, "\sqsupseteq"] & \\
A \arrow[r, "f"'] & B \arrow[r, shift left, "g'"] \arrow[r, shift right, "g"'] & C.
\end{tikzcd}
\tag{3.iv}
$$

The properties of pullbacks imply that

$$
\varphi^*_f\left(\sigma_{g'}\right) = \varphi^*_f\left(\sigma_g \cdot s\right) = \varphi^*_f\left(\sigma_g\right) \cdot s',
$$

where s' the pullback of s along $\sigma_g^*\left(\varphi_f\right)$. Moreover, the 2-cell $\varphi_{g'} \sqsubseteq \varphi_g \cdot s$ yields

$$\varphi_{g'} \cdot \sigma_{g'}^*\left(\varphi_f\right) \sqsubseteq \varphi_g \cdot s \cdot \sigma_{g'}^*\left(\varphi_f\right) = \varphi_g \cdot \sigma_g^*\left(\varphi_f\right) \cdot s'.$$

We conclude that $g' \cdot f \leq g \cdot f$. □

Before we proceed to discuss monotonicity of composition on the left we need to provide the following definitions.

Definition 3.4. *Let $\mathscr{C}$ be an* **Ord***-category. A morphism $m : A \longrightarrow B$ is*

- ***full*** *if for every $u, v : X \longrightarrow A$ such that $m \cdot u \sqsubseteq m \cdot v$, then $u \sqsubseteq v$;*
- ***upper-closed*** *if for every pair of morphisms $u : X \longrightarrow A$ and $v : X \longrightarrow B$ forming the 2-cell*

$$\begin{tikzcd} & X \arrow[dl, "u"'] \arrow[dr, "v"] \arrow[d, phantom, "\sqsubseteq"] & \\ A \arrow[rr, "m"'] & {} & B \end{tikzcd} \tag{3.v}$$

  then there exists $z : X \longrightarrow A$ such that $v = m \cdot z$.

A map, which is both full and upper-closed, can be characterised as follows.

Lemma 3.5. *A morphism $m : A \longrightarrow B$ is full and upper-closed if and only if it is monomorphic and, for every pair of morphisms $u : X \longrightarrow A$ and $v : X \longrightarrow B$ such that we have the 2-cell $m \cdot u \sqsubseteq v$, then there exists a unique morphism $z : X \longrightarrow A$ in the diagram*

$$\begin{tikzcd} & X \arrow[dl, bend right, "u"'] \arrow[dl, dashed, "z"] \arrow[dr, "v"] \arrow[d, phantom, "\circlearrowright"] & \\ A \arrow[rr, "m"'] & {} & B. \end{tikzcd}$$

Proof. $\Rightarrow$. It is trivial to notice that if m is full, then it is monomorphic. Moreover, for every pair of morphisms u and v such that $m \cdot u \sqsubseteq v$, then there exists z such that $v = m \cdot z$, since m is upper-closed. From the 2-cell $m \cdot u \sqsubseteq v = m \cdot z$ follows that $u \sqsubseteq z$, since m is full. Given another morphism z' such that $v = m \cdot z'$ and $u \sqsubseteq z'$, we have that $m \cdot z = v = m \cdot z'$ implies that $z = z'$, since m is monomorphic as proved above.

$\Leftarrow$. First we remark that the hypotheses trivially yield that m is upper-closed. Let us consider the 2-cell $m \cdot u \sqsubseteq m \cdot u'$, that compose the diagram

$$\begin{tikzcd} & X \arrow[dl, "u"'] \arrow[dr, "u'"] \arrow[d, phantom, "\sqsubseteq"] & \\ A \arrow[rr, "m"'] & {} & A \arrow[d, "m"] \\ & & B. \end{tikzcd}$$

Then there exists a unique morphism $z : X \longrightarrow A$ such that $m \cdot z = m \cdot u'$ and $u \sqsubseteq z$. Since m is monomorphic, then $z = u'$ and $u \sqsubseteq u'$. Hence m is full. $\square$

Now we can state the following result.

Theorem 3.6. *Let $\mathscr{C}$ be an* **Ord**- *category and $\mathcal{S}$ a class of admissible subobjects. If every morphism in $\mathcal{S}$ is full and upper-closed and $s', s' \cdot s \in \mathcal{S}$ implies that $s \in \mathcal{S}$, then the partial order $\leq$ yields an* **Ord**-*enrichment on $\mathcal{P}_{\mathcal{S}}(\mathscr{C})$.*

Proof. In general $\leq$ defines a partial order relation among partial morphisms with the same domain and codomain. Such partial order relation respects composition on the right by Proposition 3.3. We need to check that composition on the left is monotone as well. Let us consider $f' \leq f : Q \relbar\joinrel\mapstochar\joinrel\rightarrow B$ and $g : B \relbar\joinrel\mapstochar\joinrel\rightarrow C$. We can write the following diagram

$$
\begin{tikzcd}
 & & D_{g\cdot f'} \arrow[dl, tail, "\varphi^*_{f'}(\sigma_g)"'] \arrow[ddr, bend left, "\sigma^*_g(\varphi_{f'})"] & & \\
 & D_{f'} \arrow[d, tail, "t"] \arrow[dl, tail, bend right, "\sigma_{f'}"'] \arrow[ddr, bend left, "\varphi_{f'}"] & D_{g\cdot f} \arrow[dl, tail] \arrow[dr] & & \\
 & D_f \arrow[dl, tail, "\sigma_f"'] \arrow[dr, "\varphi_f"] \arrow[r, phantom, "\sqsubseteq"] & & D_g \arrow[dl, tail, "\sigma_g"'] \arrow[dr, "\varphi_g"] & \\
A \arrow[rr, shift left, "f'", "\shortmid" marking] \arrow[rr, shift right, "f"', "\shortmid" marking] & & B \arrow[rr, "g"', "\shortmid" marking] & & C
\end{tikzcd}
$$

In particular, the 2-cell $\varphi_{f'} \sqsubseteq \varphi_f \cdot t$ yields

$$
\sigma_g \cdot \sigma_g^*\left(\varphi_{f'}\right) = \varphi_{f'} \cdot \varphi^*_{f'}(\sigma_g) \sqsubseteq \varphi_f \cdot t \cdot \varphi^*_{f'}(\sigma_g)\,. \tag{3.vi}
$$

Since σ_g is full and upper-closed, then (3.vi) yields that there exists the morphism z such that $\sigma_g \cdot z = \varphi_f \cdot t \cdot \varphi^*_{f'}(\sigma_g)$ and $\sigma_g^*\left(\varphi_{f'}\right) \sqsubseteq z$, thus it gives rise to the universal morphism t' as in the diagram

$$
\begin{tikzcd}
 & D_{g\cdot f'} \arrow[dl, tail, "\varphi^*_{f'}(\sigma_g)"'] \arrow[d, tail, dotted, "t'"'] \arrow[ddr, dashed, "z"] \arrow[ddr, bend left=40, "\sigma^*_g(\varphi_{f'})"] & \\
D_{f'} \arrow[d, tail, "t"'] & D_{g\cdot f} \arrow[dl, tail, "\varphi^*_f(\sigma_g)"] \arrow[dr, "\sigma^*_g(\varphi_f)"'] & \\
D_f \arrow[dr, "\varphi_f"'] & & D_g \arrow[dl, tail, "\sigma_g"] \\
 & B. &
\end{tikzcd} \tag{3.vii}
$$

Since $\varphi_f^*(\sigma_g) \cdot t' = t \cdot \varphi^*_{f'}(\sigma_g)$ belongs to $\mathcal{S}$, then t' belongs to $\mathcal{S}$ as well. This allows us to conclude

$$
\begin{cases}
\sigma_f \cdot \varphi_f^*(\sigma_g) \cdot t' = \sigma_f \cdot t \cdot \varphi^*_{f'}(\sigma_g) = \sigma_{f'} \cdot \varphi^*_{f'}(\sigma_g) \\
\varphi_g \cdot \sigma_g^*\left(\varphi_{f'}\right) \sqsubseteq \varphi_g \cdot z = \varphi_g \cdot \sigma_g^*\left(\varphi_f\right) \cdot t',
\end{cases}
$$

which yields that $f' \cdot g \leq f \cdot g$. □

Remark 3.7. *The previous theorem may be applied in particular to the partial order $\leq$ yielded by the discrete* **Ord***-enrichment on $\mathscr{C}$. This is true since all admissible subobjects are monomorphic and 2-cells as* (3.v) *are equalities satisfying the equivalent condition of Lemma 3.5.*

Notation 3.8. *We have proved that composition to the left and to the right is monotone and thus $\leq$ gives rise to an* **Ord***-enrichment on $\mathcal{P}(\mathscr{C})$. Hence, given $f' \leq f$ and $g' \leq g$, then $g' \cdot f' \leq g \cdot f$. This is true and can be observed by steps as*

$$g' \cdot f' \leq g' \cdot f \leq g \cdot f.$$

The admissible subobject morphism that witnesses the existence of the 2-cell $g' \cdot f' \leq g \cdot f$ is defined applying subsequently the constructions shown in the proofs of Proposition 3.3 and Theorem 3.6. Hinting to the notation of Definition 2.41, we will denote such morphism by $t * s : D_{g' \cdot f'} \longrightarrow D_{g \cdot f}$, where $t : D_{f'} \longrightarrow D_f$ and $s : D_{g'} \longrightarrow D_g$ are the morphisms that define $f' \leq f$ and $g' \leq g$. We will also denote s' as in (3.iv) *by $s * f$ and t' as in* (3.vii) *by $g * t$.*

Proposition 3.9. *Let $\mathscr{C}$ be an* **Ord***-category. Let $\mathcal{P}(\mathscr{C})$ be a category of partial maps equipped with the* **Ord***-enrichment induced by the one on $\mathscr{C}$. If f is total and $f \leq g$, then g is total. Moreover, if the partial order is induced by the discrete one, then $f = g$. Hence total maps are maximal elements in their own Hom-Sets.*

Proof. If $f \leq g$ is a 2-cell, then diagram (3.iii) shows that $D_f \leq D_g$ as subobject, by the existence of the subobject morphism $s : D_f \longrightarrow D_g$. Therefore, if $D_f = A$, then $D_g = A$ as well. Furthermore, this yields that s is an isomorphism, hence if the **Ord**-enrichment on $\mathscr{C}$ is discrete, then the two spans are equivalent and $f = g$. □

3.2.1 Adjunctions between partial maps

Since adjoint morphisms play an important role for lax factorisations, as we will see in the next chapter, we focus now on describing adjoint morphisms for categories of partial maps. We will provide a result that completely characterises adjoint pairs of morphisms for any **Ord**-enrichment on $\mathcal{P}(\mathscr{C})$. Then we discuss the link between adjunction in $\mathcal{P}(\mathscr{C})$ and adjunction in $\mathscr{C}$.

We briefly state some terminology that we will use along the thesis. In an **Ord**-enriched category, given an adjunction $f \dashv g$, we will refer to $\mathrm{id}_A \leq g \cdot f$ as the *unit* 2-cell and to $f \cdot g \leq \mathrm{id}_B$ as the *counit* 2-cell.

Proposition 3.10. *Let $\mathcal{P}(\mathscr{C})$ be a category of partial maps with $\mathscr{C}$ an* **Ord***-enriched category. A pair of morphisms in $\mathcal{P}(\mathscr{C})$ constitute an adjunction $f \dashv g$ if and only if f is total and $\varphi_f = \sigma_g \cdot \gamma$, for a γ such that $\gamma \dashv \varphi_g$ in $\mathscr{C}$.*

Proof. First we consider the adjunction $f \dashv g$ in $\mathcal{P}(\mathscr{C})$. By Proposition 3.9, the 2-cell $\mathrm{id}_A \leq g \cdot f$ yields that $g \cdot f$ is total and therefore f is total by Proposition 3.1. We write explicitly the 2-cells

$$
\begin{tikzcd}
A \arrow[d, tail] \arrow[dr, "\gamma"] \arrow[dd, bend right=60, "\mathrm{id}_A"'] \arrow[ddrr, bend left=60, "\mathrm{id}_A"] \arrow[drr, phantom, "\sqsubseteq"] & & \\
A \arrow[d, tail] \arrow[dr, "\varphi_f"] & D_g \arrow[d, tail, "\sigma_g"] \arrow[dr, "\varphi_g"] & \\
A \arrow[r, "f"', "+" marking] & B \arrow[r, "g"', "+" marking] & A.
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[ddd, bend right=30, "\mathrm{id}_B"'] \arrow[dddrrr, bend left=40, "\mathrm{id}_B"] & & & \\
& D_g \arrow[ul, tail, "\sigma_g"'] \arrow[d, tail] \arrow[dr, "\varphi_g"] \arrow[ur, phantom, "\sqsubseteq"] & & \\
& D_g \arrow[d, tail, "\sigma_g"] \arrow[dr, "\varphi_g"] & A \arrow[d, tail] \arrow[dr, "\varphi_f"] & \\
& B \arrow[r, "g"', "+" marking] & A \arrow[r, "f"', "+" marking] & B.
\end{tikzcd}
\tag{3.viii}
$$

The subobject morphism of the left diagram is id_A due to the stated totality of f and $g \cdot f$. We notice immediately that $\varphi_f = \sigma_g \cdot \gamma$, where γ is the pullback of φ_f along σ_g. We prove that γ is the morphism sought. We have the 2-cells $\mathrm{id}_A \sqsubseteq \varphi_g \cdot \gamma$ and $\sigma_g \cdot \gamma \cdot \varphi_g = \varphi_f \cdot \varphi_g \sqsubseteq \sigma_g$. Since by assumption $\sigma_g \in \mathcal{S}$ is full, we deduce that $\gamma \cdot \varphi_g \sqsubseteq \mathrm{id}_B$. This yields the existence of the adjunction $\gamma \dashv \varphi_g$ in $\mathscr{C}$.

The other direction is proved simply by calculation. In fact, since $\varphi_f = \sigma_g \cdot \gamma$, then the pullback of φ_f along σ_g is γ. This allows us to write the left diagram in (3.viii), where the 2-cell $\mathrm{id}_A \sqsubseteq \varphi_g \cdot \gamma$ is the unit 2-cell of the adjunction $\gamma \dashv \varphi_g$. If we compose the counit 2-cell $\gamma \cdot \varphi_g \sqsubseteq \mathrm{id}_{D_g}$ with σ_g, we obtain $\varphi_f = \sigma_g \cdot \gamma \cdot \varphi_g \sqsubseteq \sigma_g$, which allows us to write the right diagram. In conclusion this yields the adjunction $f \dashv g$ in $\mathcal{P}(\mathscr{C})$. □

From here on, whenever we use this characterization, we denote the morphism γ in the statement by $\widetilde{\varphi}_f$.

If we consider an ordinary category $\mathscr{C}$, we can state the following corollary.

Corollary 3.11. *Let $\mathcal{P}(\mathscr{C})$ be a category of partial morphisms equipped with the* **Ord***-enrichment induced by the ordinary category $\mathscr{C}$. A pair of morphisms in $\mathcal{P}(\mathscr{C})$ constitute an adjunction $f \dashv g$ if and only if $f = (\mathrm{id}, \sigma)$ and $g = (\sigma, \mathrm{id})$, for some $\sigma \in \mathcal{S}$.*

This is due to the fact that pairs of adjoint morphisms in a discrete category are pairs of inverse morphisms.

We conclude this paragraph providing an overview on how the previous results link adjoint morphisms between $\mathscr{C}$ and $\mathcal{P}(\mathscr{C})$. Let $\mathscr{C}$ be an **Ord**-category and $\mathcal{P}(\mathscr{C})$ equipped with the **Ord**-enrichment inherited by $\mathscr{C}$. We denote by $\mathsf{Adj}(\mathscr{C})$ the class of adjoint pairs of morphisms in $\mathscr{C}$ and by $\mathsf{Adj}(\mathcal{P}(\mathscr{C}))$ the class of those in $\mathcal{P}(\mathscr{C})$. Then $\mathsf{Adj}(\mathcal{P}(\mathscr{C}))$ is equipped with a partial order. We have that $(f \dashv g) \leq (f' \dashv g')$ if and only if there exists $s \in \mathcal{S}$ such that $\varphi_f = s \cdot \varphi_{f'}$ and $\sigma_g = s \cdot \sigma_{g'}$. We consider the following map

$$
\begin{aligned}
I : \mathsf{Adj}(\mathscr{C}) &\longrightarrow \mathsf{Adj}(\mathcal{P}(\mathscr{C})) \\
(f \dashv g) &\longrightarrow ((\mathrm{id}_A, f) \dashv (\mathrm{id}_B, g))\,.
\end{aligned}
$$

If $\mathsf{Adj}(\mathscr{C})$ is equipped with the discrete partial order, then I is an injective monotone map. Furthermore, Proposition 3.10 allows us to build the following map

$$\begin{aligned} J : \mathsf{Adj}(\mathcal{P}(\mathscr{C})) &\longrightarrow \mathsf{Adj}(\mathscr{C}) \\ \big((\mathrm{id}_A, \varphi_f) \dashv (\sigma_g, \varphi_g)\big) &\longrightarrow \big(\widetilde{\varphi}_f \dashv \varphi_g\big). \end{aligned}$$

If $(f \dashv g) \leq (f' \dashv g')$, then there exists $s \in \mathcal{S}$ such that $(f \dashv g) = \big((\mathrm{id}_A, s \cdot \sigma_{g'} \cdot \widetilde{\varphi}_f) \dashv (s \cdot \sigma_{g'}, \varphi_g)\big)$, hence $J(f \dashv g) = \big(\widetilde{\varphi}_f \dashv \varphi_g\big) = J(f' \dashv g')$, hence J is monotone. Then it is straightforward that

$$JI\big(\varphi_f \dashv \varphi_g\big) = J\big((\mathrm{id}_A, \varphi_f) \dashv (\mathrm{id}_B, \varphi_g)\big) = \big(\varphi_f \dashv \varphi_g\big),$$

therefore $JI = \mathrm{Id}_{\mathsf{Adj}(\mathscr{C})}$.

Moreover, we consider $IJ(f \dashv g) = \big((\mathrm{id}_A, \widetilde{\varphi}_f) \dashv (\mathrm{id}_B, \varphi_B)\big)$, then σ_g yields

$$\big((\mathrm{id}_A, \varphi_f) \dashv (\sigma_g, \varphi_g)\big) \leq \big((\mathrm{id}_A, \widetilde{\varphi}_f) \dashv (\mathrm{id}_B, \varphi_g)\big).$$

In conclusion we have

$$\begin{cases} \mathrm{Id}_{\mathsf{Adj}(\mathcal{P}(\mathscr{C}))} \leq IJ \\ JI = \mathrm{Id}_{\mathsf{Adj}(\mathscr{C})} \end{cases}$$

This yields an adjunction between the partially ordered sets $\mathsf{Adj}(\mathcal{P}(\mathscr{C}))$ and $\mathsf{Adj}(\mathscr{C})$.

Chapter 4

Lax factorisation systems

In this chapter we present the theoretical contribution of this work which was introduced in [Lar21]. We introduce a new notion of orthogonality for **Ord**-enriched categories and investigate the factorisation systems that arise as a consequence. We will propose definitions and properties on the blueprint of traditional factorisation systems that we recalled in Chapter 2. In particular we will introduce and analyse a lax version of WFS, FFS and AWFS.

The main setting of this chapter will be **Ord**-enriched categories. Before getting into the discussion of lax factorisation systems, we recall some useful definitions and fix some notation that we will use through this chapter.

Given an **Ord**-category $\mathscr{C}$, we denote by $\mathscr{C}^2_{\text{lax}}$ the category whose objects are morphisms in $\mathscr{C}$ and morphisms are squares $(u,v) : f \longrightarrow g$ of the type

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq"] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D. \end{tikzcd} \tag{4.i}$$

We will refer to these squares as *lax squares*.

We would like to recall also the definition of comma objects.

Definition 4.1. *Given a cospan (f,g) in an* **Ord***-category $\mathscr{C}$, a* ***comma object****, or lax pullback, is the span $\left(g^{\leftarrow}(f), f^{\downarrow}(g)\right)$ forming the lax square*

$$\begin{tikzcd} f \downarrow g \arrow[r, "f^{\downarrow}(g)"] \arrow[d, "g^{\leftarrow}(f)"'] \arrow[dr, phantom, "\leq"] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B, \end{tikzcd} \tag{4.ii}$$

such that for every other span (p,q) forming the 2-cell $f\cdot q \le g\cdot p$, there exists a universal morphism $\omega_{(p,q)}$ as in the diagram

$$
\begin{tikzcd}
W \arrow[drr, bend left, "q"] \arrow[ddr, bend right, "p"'] \arrow[dr, dashed, "\omega_{(p,q)}"] & & \\
& f\downarrow g \arrow[r, "f^{\downarrow}(g)"] \arrow[d, "g^{\leftarrow}(f)"'] \arrow[dr, phantom, "\leq"] & A \arrow[d, "f"] \\
& C \arrow[r, "g"'] & B,
\end{tikzcd}
$$

where the two triangles are commutative. Moreover, for any pair of spans (p,q) and (p',q') such that $f\cdot q \le g\cdot p$, $f\cdot q' \le g\cdot p'$ and such that $p \le p'$ and $q \le q'$, then $\omega_{(p,q)} \le \omega_{(p',q')}$.

Cocomma objects are defined dually for spans and we denote them by

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq"] & C \arrow[d, "g_{\leftarrow}(f)"] \\
B \arrow[r, "f_{\downarrow}(g)"'] & f\uparrow g.
\end{tikzcd}
\tag{4.iii}
$$

We stress that, unlike regular pullbacks and pushouts, (co)comma objects are directioned, thus $f\downarrow g$ and $g\downarrow f$ are not isomorphic in general. For this reason we signal that, whenever we refer to **stability under (co)comma object** of a class of morphisms $\mathcal{H}$, we mean it in the horizontal sense, i.e. considering diagram (4.ii) or diagram (4.iii) if f belongs to $\mathcal{H}$, then $g^{\leftarrow}(f) \in \mathcal{H}$ or $g_{\leftarrow}(f) \in \mathcal{H}$ respectively.

We also recall the definition of 2-dimensional coproducts.

Definition 4.2. *Let $\mathscr{C}$ be an* **Ord***-category. The coproduct of two objects $(A \sqcup B, i_A, i_B)$ is a* ***2-dimensional coproduct*** *if i_A and i_B are jointly full.*

Moreover, if the 2-dimensional coproducts $A \sqcup A'$ and $B \sqcup B'$ exist, we can define 2-dimensional coproducts of morphisms f and g as in the following diagram

$$
\begin{tikzcd}
& A \sqcup A' \arrow[dd, dotted, "f \sqcup g"] & \\
A \arrow[ur, "i_A"] \arrow[d, "f"'] & & A' \arrow[ul, "i_{A'}"'] \arrow[d, "g"] \\
B \arrow[dr, "i_B"'] & & B' \arrow[dl, "i_{B'}"] \\
& B \sqcup B', &
\end{tikzcd}
$$

where $f \sqcup g$ is the universal morphism $[i_B\cdot f, i_{B'}\cdot g]$ related to the 2-dimensional coproduct $A \sqcup A'$.

4.1 Lax weak factorisation systems

In this section we introduce a new notion of orthogonality which is the key ingredient of the following sections. Such orthogonality relation gives rise to lax diagonal morphisms for lax squares. Then we will analyse some of its properties and thereafter we will use this new definition to introduce a lax version of WFS.

Definition 4.3. *Let $\mathscr{C}$ be an* **Ord**-*enriched category. Two morphisms in $\mathscr{C}$ are said to be* ***laxly weakly orthogonal****, denoted by $f \pitchfork g$, if, for every lax square $(u,v) : f \longrightarrow g$, there exists a morphism d such that*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq", pos=0.25] & C \arrow[d, "g"] \\ B \arrow[ur, "d"] \arrow[r, "v"'] \arrow[ur, phantom, "\leq", pos=0.75, shift right=3] & D \end{tikzcd} \qquad \begin{cases} u \leq d \cdot f \\ g \cdot d \leq v. \end{cases} \tag{4.iv}$$

We refer to d as ***lax diagonal morphism*** *or lax diagonal lifting.*

We first observe that this constitutes a generalisation of weak orthogonality, as described in Section 2.1. Indeed, whenever the partial order in $\mathscr{C}$ is discrete, the two definitions coincide trivially.

Then, the first natural step is to identify those morphisms which are laxly weak orthogonal with respect to any other morphism in the category. Before doing so, we recall that two morphisms $f : A \to B$ and $g : B \to A$ in an **Ord**-category constitute an adjunction $f \dashv g$ if we have $\mathrm{id}_A \leq g \cdot f$ and $f \cdot g \leq \mathrm{id}_B$.

Proposition 4.4. *Given a morphism $f \in \mathscr{C}^2_{\mathrm{lax}}$, the following properties are equivalent:*

1. *$f \pitchfork f$;*
2. *f is a left adjoint morphism to some f^*;*
3. *$f \pitchfork \mathscr{C}^2_{\mathrm{lax}}$;*
4. *$\mathscr{C}^2_{\mathrm{lax}} \pitchfork f$.*

Proof. 1.⇒2. Given a morphism f that is laxly weakly orthogonal to itself, we may consider the identity lax square $(\mathrm{id}_A, \mathrm{id}_B : f \longrightarrow f)$ (which is actually commutative). Then there exists the lax diagonal lifting δ in the following diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq", pos=0.25] & A \arrow[d, "f"] \\ B \arrow[ur, dotted, "\delta"'] \arrow[r, "\mathrm{id}_B"'] \arrow[ur, phantom, "\leq", pos=0.75, shift right=3] & B \end{tikzcd}$$

It follows that the diagonal morphism δ is the right adjoint and the two 2-cells in the two triangles define the adjunction sought.

2.⇒3. Let $(u,v): f \longrightarrow g$ be a lax square. We consider $\delta = u \cdot f^*$. Then the adjunction yields

$$\begin{cases} u \leq u \cdot f^* \cdot f = \delta \cdot f \\ g \cdot \delta = g \cdot u \cdot f^* \leq v \cdot f \cdot f^* \leq v. \end{cases} \tag{4.v}$$

Thus δ is a lax diagonal morphism for the lax square (u,v).

2.⇒4. Analogously, given any lax square $(u,v): g \longrightarrow f$, we have that $\delta = f^* \cdot v$ is a lax diagonal filler.

3.⇒1. and 4.⇒1. are straightforward. □

For any class of morphisms $\mathcal{H}$ we can define its lax weak orthogonal complements as follows

$$^{\pitchfork}\mathcal{H} = \{f | f \pitchfork h \text{ for every } h \in \mathcal{H}\} \qquad \text{and} \qquad \mathcal{H}^{\pitchfork} = \{f | h \pitchfork f \text{ for every } h \in \mathcal{H}\}.$$

Then we have that lax orthogonal complements carry the following properties.

Proposition 4.5. *Let $\mathscr{C}$ be an* **Ord***-category. The pair $^{\pitchfork}(-)$ and $(-)^{\pitchfork}$ forms a Galois connection among the classes of morphisms in $\mathscr{C}$ partially ordered by the inclusion.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms in $\mathscr{C}$. We prove first that $^{\pitchfork}(-)$ and $(-)^{\pitchfork}$ are antitone with respect to inclusion. Let $\mathcal{A} \subseteq \mathcal{B}$. If $f \in {^{\pitchfork}\mathcal{B}}$, then $f \pitchfork b$ for every $b \in \mathcal{B}$, and in particular $f \pitchfork a$ for every $a \in \mathcal{A}$, thus $f \in {^{\pitchfork}\mathcal{A}}$. Therefore $^{\pitchfork}\mathcal{A} \supseteq {^{\pitchfork}\mathcal{B}}$. Similarly one proves that $\mathcal{A}^{\pitchfork} \supseteq \mathcal{B}^{\pitchfork}$.

Moreover, $\mathcal{A} \subseteq {^{\pitchfork}\mathcal{B}}$ if and only if for every $a \in \mathcal{A}$ and for every $b \in \mathcal{B}$, $a \pitchfork b$, which is equivalent to $\mathcal{B} \subseteq {^{\pitchfork}\mathcal{A}}$. Analogously $\mathcal{A} \subseteq \mathcal{B}^{\pitchfork}$ is equivalent to $\mathcal{B} \subseteq \mathcal{A}^{\pitchfork}$. □

In particular this proposition yields the following corollary.

Corollary 4.6. *Let $\mathcal{H}$ be a class of morphisms in $\mathscr{C}$. Then $\mathcal{H} \subseteq {^{\pitchfork}(\mathcal{H}^{\pitchfork})}$ and $\mathcal{H} \subseteq ({^{\pitchfork}\mathcal{H}})^{\pitchfork}$. Moreover, $\mathcal{H}^{\pitchfork} = ({^{\pitchfork}(\mathcal{H}^{\pitchfork})})^{\pitchfork}$ and $^{\pitchfork}\mathcal{H} = {^{\pitchfork}}\left(({^{\pitchfork}\mathcal{H}})^{\pitchfork}\right)$.*

We report that the proofs of the last two results are a direct translation of the proof of Proposition 2.2 and Corollary 2.3.

Proposition 4.7. *Let $\mathcal{H}$ be a class of morphisms in an* **Ord***-category $\mathscr{C}$. Then the following assertions hold.*

1. *$\mathcal{H}^{\pitchfork}$ and $^{\pitchfork}\mathcal{H}$ contain all the left adjoint morphisms of $\mathscr{C}$;*

2. *$\mathcal{H}^{\pitchfork}$ and $^{\pitchfork}\mathcal{H}$ are closed under composition.*

Proof. 1. This is a consequence of Proposition 4.4.

2. We show this by building subsequent diagonal liftings. Let $f, g \in \mathcal{H}^{\pitchfork}$ be two composable arrows. Then for every lax square $(u,v) : h \longrightarrow g \cdot f$, with $h \in \mathcal{H}$, we can build the following diagram

$$\begin{tikzcd}
H \arrow[r, "u"] \arrow[dd, "h"'] & A \arrow[d, "f"] \\
 & B \arrow[d, "g"] \\
H' \arrow[r, "v"'] \arrow[ruu, dotted, "\delta_{g\cdot f}" description] \arrow[ru, dotted, "\delta_g" description] & C;
\end{tikzcd}$$

where $\delta_{g\cdot f}$ is a lax diagonal lifting for the square (u, δ_g). Then we have

$$g \cdot f \cdot \delta_{g\cdot f} \leq g \cdot \delta_g \leq v.$$

We conclude that $\delta_{g\cdot f}$ is a diagonal lifting for (u,v). Thus $g \cdot f \in \mathcal{H}^{\pitchfork}$. The proof for ${}^{\pitchfork}\mathcal{H}$ is similar. $\square$

Proposition 4.8. *Let $\mathcal{H}$ be a class of morphisms in an* **Ord***-category $\mathscr{C}$. Then $\mathcal{H}^{\pitchfork}$ is closed under comma objects.*

Proof. We consider $f \in \mathcal{H}^{\pitchfork}$ and the comma object

$$\begin{tikzcd}
f \downarrow g \arrow[r, "f^{\downarrow}(g)"] \arrow[d, "g^{\leftarrow}(f)"'] \arrow[dr, phantom, "\geq"] & A \arrow[d, "f"] \\
C \arrow[r, "g"'] & B.
\end{tikzcd} \tag{4.vi}$$

Let $(u,v) : h \longrightarrow g^{\leftarrow}(f)$ be a lax square with $h \in \mathcal{H}$. Then, there exists a lax diagonal morphism δ lifting h against f, as shown in the following diagram

$$\begin{tikzcd}
H \arrow[r, "u"] \arrow[d, "h"'] \arrow[dr, phantom, "\geq"] & f \downarrow g \arrow[r, "f^{\downarrow}(g)"] \arrow[d, "g^{\leftarrow}(f)"] \arrow[dr, phantom, "\geq"] & A \arrow[d, "f"] \\
H' \arrow[r, "v"'] \arrow[rru, dotted, "\delta" description] & C \arrow[r, "g"'] & B.
\end{tikzcd} \tag{4.vii}$$

Hence we have the 2-cell $f\cdot\delta\leq g\cdot v$, and therefore, by the properties of comma objects, there exists the universal morphism δ' in the following diagram

$$
\begin{tikzcd}
H' \arrow[drr, dashed, "\delta'"] \arrow[drrr, bend left, "\delta"] \arrow[ddrr, bend right, "v"'] & & & \\
& & f\downarrow g \arrow[r, "f^{\downarrow}(g)"] \arrow[d, "g^{\leftarrow}(f)"'] \arrow[dr, phantom, "\leq"] & A \arrow[d, "f"] \\
& & C \arrow[r, "g"'] & B.
\end{tikzcd}
$$

In particular, we obtain $g^{\leftarrow}(f)\cdot\delta'=v$. Furthermore, we can write the following diagram

$$
\begin{tikzcd}
H \arrow[drr, shift left, "u"] \arrow[drr, shift right, "\delta'\cdot h"'] \arrow[drr, phantom, "\leq"] \arrow[drrr, bend left, shift left, "f^{\downarrow}(g)\cdot u"] \arrow[drrr, bend left, shift right, "\delta\cdot h"'] \arrow[ddrr, bend right, shift left, "g^{\leftarrow}(f)\cdot u"] \arrow[ddrr, bend right, shift right, "v\cdot h"'] & & & \\
& & f\downarrow g \arrow[r, "f^{\downarrow}(g)"] \arrow[d, "g^{\leftarrow}(f)"'] \arrow[dr, phantom, "\leq"] & A \arrow[d, "f"] \\
& & C \arrow[r, "g"'] & B,
\end{tikzcd}
$$

where the 2-cell $u\leq\delta'\cdot h$ exists by the properties of comma objects. Thus, we have checked that $u\leq\delta'\cdot h$ and $g^{\leftarrow}(f)\cdot\delta'=v$. Hence δ' is a lax diagonal morphism for the lax square (u,v) and therefore $g^{\leftarrow}(f)\in\mathcal{H}^{\pitchfork}$. $\square$

The last result yields the following dual proposition.

Proposition 4.9. *Let $\mathcal{H}$ be a class of morphisms in an* **Ord**-*category $\mathscr{C}$. Then ${}^{\pitchfork}\mathcal{H}$ is closed under cocomma objects.*

Proposition 4.10. *Let $\mathcal{H}$ be a class of morphisms of a category $\mathscr{C}$ with 2-dimensional coproducts. The class ${}^{\pitchfork}\mathcal{H}$ is closed under 2-dimensional coproducts.*

Proof. We consider $\hat{h}=\coprod_{i\in I}h_i:\coprod_{i\in I}\longrightarrow\coprod_{i\in I}B_i$, such that $h_i\in{}^{\pitchfork}\mathcal{H}$ for every $i\in I$. For each lax square $(u,v):\hat{h}\longrightarrow g$, with $g\in\mathcal{H}$ we have

$$
\begin{tikzcd}
A_i \arrow[r, "\sigma_i"] \arrow[d, "h_i"'] & \coprod_{i\in I}A_i \arrow[r, "u"] \arrow[d, "\hat{h}"] \arrow[dr, phantom, "\leq"] & X \arrow[d, "g"] \\
B_i \arrow[r, "\sigma_i'"] & \coprod_{i\in I}B_i \arrow[r, "v"] & B.
\end{tikzcd}
$$

Since $h_i \in {}^{\pitchfork}\mathcal{H}$ for each $i \in I$, then we get a family $(\delta_i : B_i \longrightarrow X)_{i\in I}$ of lax diagonal morphisms lifting h_i against g. By the definition of 2-dimensional coproduct, there exists the universal arrow $\delta = [\delta_i]_{i\in I} : \coprod_{i\in I} B_i \longrightarrow X$ and since $(\sigma_i)_{i\in I}$ and $(\sigma_i')_{i\in I}$ are jointly full, it implies that δ is a lax diagonal morphism lifting $\hat{h}$ against g. Therefore $\hat{h} \in {}^{\pitchfork}\mathcal{H}$. □

Again we may state the following dual result.

Proposition 4.11. *Let $\mathcal{H}$ be a class of morphisms of a category $\mathscr{C}$ with 2-dimensional products. The class $\mathcal{H}^{\pitchfork}$ is closed under 2-dimensional products.*

We can now introduce the definition of lax weak factorisation system.

Definition 4.12. *A* ***lax weak prefactorisation system*** *is a pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ such that $\mathcal{R} = \mathcal{L}^{\pitchfork}$ and $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$. Moreover, if any morphism $f \in \mathscr{C}^2_{\text{lax}}$ has an $(\mathcal{L}, \mathcal{R})$-factorisation*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathcal{L}\ni l_f"'] & & B \\ & W_f \arrow[ur, "r_f\in\mathcal{R}"'] & \end{tikzcd} \tag{4.viii}$$

then $(\mathcal{L}, \mathcal{R})$ is said to be a ***lax weak factorisation system*** *(WFS_{lax}).*

We remark that, given a (WFS_{lax}) $(\mathcal{L}, \mathcal{R})$, for any lax square $(u, v) : f \longrightarrow g$ there exists a morphism δ as in the diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "l_f"'] \arrow[dr, phantom, "\nearrow"] & C \arrow[d, "l_g"] \\ W_f \arrow[r, dashed, "\delta"] \arrow[d, "r_f"'] \arrow[dr, phantom, "\nearrow"] & W_g \arrow[d, "r_g"] \\ B \arrow[r, "v"'] & D. \end{tikzcd} \tag{4.ix}$$

The morphism δ is obtained by the lax weak orthogonality relation $l_f \pitchfork r_g$.

Furthermore, as a consequence of Proposition 4.4, we have that left adjoint morphisms belong to any lax weak orthogonal complement and they constitute the intersection between the two classes of morphisms of any lax weak prefactorisation system.

We also point out that uniqueness of such lax diagonal liftings is not granted in general. In fact, given a morphism f satisfying the conditions of Proposition 4.4, we have that in any lax square $(u, v) : f \longrightarrow f$ both $d_u = u \cdot f^*$ and $d_v = f^* \cdot v$ are suitable lax diagonal morphisms.

As a consequence of Proposition 4.5 and Corollary 4.6, we have the following result which enables us to build lax weak prefactorisation systems from any class of morphisms.

Proposition 4.13. *Given a class of morphisms $\mathcal{H} \subseteq \mathscr{C}^2$, then $\left({}^{\pitchfork}(\mathcal{H}^{\pitchfork}); \mathcal{H}^{\pitchfork}\right)$ and $\left({}^{\pitchfork}\mathcal{H}; ({}^{\pitchfork}\mathcal{H})^{\pitchfork}\right)$ are lax prefactorisation systems.*

Remark 4.14. *Before proceeding with the discussion of lax factorisation systems, we would like to compare the definitions given in this section and the enriched factorisation systems that we have mentioned in Section 2.7. Let us consider diagram* (2.xiv) *in Definition 2.50. Then one notices that*

$$\delta : \mathscr{C}^2(f,g) \longrightarrow \mathscr{K}(B,C)$$

maps each commutative square to a diagonal morphism, but does not take into consideration lax squares. Another difference is that, as stated in [Rie14], δ constitutes a functorial association between commutative squares and diagonal morphism. Instead, $\mathrm{WFSs}_{\mathrm{lax}}$ *admit a functorial realisation under some conditions that we will explore in Section 4.3.*

4.2 Small object argument for lax weak factorisation systems

In this section we aim to prove an analogous version for the lax context of the *Small object argument* that we have presented in Subsection 2.2. This result is particularly useful since it gives a way to build factorisation systems starting from a set of maps under some cocompleteness and smallness conditions through a transfinite construction.

Before describing this result for $\mathrm{WFSs}_{\mathrm{lax}}$, we need to adapt the definitions we use to state our smallness conditions.

Definition 4.15. *Let $\mathscr{C}$ be an* **Ord***-category and λ be an ordinal. By* ***lax colimit cocone*** *of a diagram*

$$\left(e^0_\alpha : X_0 \longrightarrow X_\alpha\right)_{\alpha<\lambda}$$

we mean a jointly full family of morphisms $\left(e^\alpha_\lambda : X_\alpha \longrightarrow X_\lambda\right)_{\alpha<\lambda}$, such that $e^0_\lambda \leq e^\alpha_\lambda \cdot e^0_\alpha$ and for every family of morphisms $(q^\alpha : X_\alpha \longrightarrow Q)_{\alpha<\lambda}$, such that $q^0 \leq q^\alpha \cdot e^0_\alpha$ for every $\alpha < \lambda$, there exists a morphism q as in the diagram

$$\begin{tikzcd}
X_0 \arrow[rr, "e^0_\alpha"] \arrow[dr, "e^0_\lambda"'] \arrow[ddr, bend right, "q^0"'] & {} \arrow[d, phantom, "\leq"] & X_\alpha \arrow[dl, "e^\alpha_\lambda"] \arrow[ddl, bend left, "q^\alpha"] \\
& X_\lambda \arrow[d, "q"] & \\
& Q, &
\end{tikzcd}$$

such that $q^\alpha = q \cdot e^\alpha_\lambda$ for each $\alpha < \lambda$. Moreover, we will refer to the morphism e^0_λ as the ***transfinite lax composition*** *of $\left(e^0_\alpha\right)_{\alpha<\lambda}$.*

We proceed proving the following proposition.

Proposition 4.16. *Let $\mathscr{C}$ be an* **Ord***-category and $\mathcal{H}$ a class of morphism in $\mathscr{C}$. Then ${}^{\pitchfork}\mathcal{H}$ is closed under transfinite lax compositions.*

Proof. We consider a diagram $\left(e^0_\alpha : X_0 \longrightarrow X_\alpha\right)_{\alpha<\lambda}$ and the lax colimit cocone $\left(e^\alpha_\lambda : X_\alpha \longrightarrow X_\lambda\right)_{\alpha<\lambda}$ for such diagram. We aim to prove that e^0_λ belongs to ${}^{\pitchfork}\mathcal{H}$. We consider any lax square $(u,v) : e^0_\lambda \longrightarrow f$, with $f \in \mathcal{H}$. We can write the following diagram

$$\begin{tikzcd}
X_0 \arrow[rr, "u"] \arrow[dr, "e^0_\alpha"] \arrow[dd, "e^0_\lambda"'] & {} \arrow[d, phantom, "\leq"] & A \arrow[dd, "f"] \\
{} \arrow[r, phantom, "\leq"] & X_\alpha \arrow[ur, dashed, "d_\alpha"] \arrow[dl, "e^\alpha_\lambda"] & {} \arrow[d, phantom, "\leq"] \\
X_\lambda \arrow[rr, "v"'] \arrow[uurr, dotted, bend right, "d"'] & & B.
\end{tikzcd}$$

We know that $f \cdot u \leq v \cdot e^0_\lambda \leq \left(v \cdot e^\alpha_\lambda\right) \cdot e^0_\alpha$ and every morphism e^0_α belongs to ${}^{\pitchfork}\mathcal{H}$, for any $\alpha < \lambda$. Hence we obtain the morphisms d_α as lax diagonal liftings of e^0_α against f. This gives rise to the family of morphisms $(d_\alpha : X_\alpha \longrightarrow A)_{\alpha<\lambda}$ such that $u = d_0 \leq d_\alpha \cdot e^0_\alpha$. This yields the universal morphism in the diagram

$$\begin{tikzcd}
X_0 \arrow[rr, "e^0_\alpha"] \arrow[dr, "e^0_\lambda"'] \arrow[ddr, bend right, "d_0"'] & {} \arrow[d, phantom, "\leq"] & X_\alpha \arrow[dl, "e^\alpha_\lambda"] \arrow[ddl, bend left, "d_\alpha"] \\
& X_\lambda \arrow[d, "d"] & \\
& A. &
\end{tikzcd}$$

This yields $d \cdot e^0_\lambda = d_0 = u$. Moreover, we have that $f \cdot d \cdot e^\alpha_\lambda = f \cdot d_\alpha \leq v \cdot e^\alpha_\lambda$ and since $\left(e^\alpha_\lambda\right)_{\alpha<\lambda}$ are jointly full, this allows us to conclude that $f \cdot d \leq v$ and therefore $e^0_\lambda \in {}^{\pitchfork}\mathcal{H}$. □

Definition 4.17. *Let λ be an ordinal. A **lax $\mathcal{H}$-cell λ-complex** is a diagram $\left(e^\alpha_\beta : E_\alpha \longrightarrow E_\beta\right)_{\alpha\leq\beta<\lambda}$ such that $e^0_\beta \leq e^\alpha_\beta \cdot e^0_\alpha$ for any $\alpha \leq \beta < \lambda$ and each morphism e^0_α is a transfinite composition of 2-dimensional coproducts of morphisms in ${}^{\pitchfork}\left(\mathcal{H}^{\pitchfork}\right)$ and morphisms obtained from elements of $\mathcal{H}$ via cocomma objects. We will refer to a lax $\mathcal{H}$-cell λ-complex $\left(e^\alpha_\beta : E_\alpha \longrightarrow E_\beta\right)_{\alpha\leq\beta\leq\lambda}$ that extends to λ as **lax $\mathcal{H}$-cell $\overline{\lambda}$-complex**.*

Definition 4.18. *Let $\mathscr{C}$ be an* **Ord**-*category, $\mathcal{H}$ a set of morphisms and $\mathscr{C}$ has transfinite lax compositions of lax $\mathcal{H}$-cell $\overline{\lambda}$-complexes. An object W is **laxly small relative to** $\mathcal{H}$ if there exists an ordinal κ such that for every $\lambda > \kappa$, any lax $\mathcal{H}$-cell λ-complex $\left(e^\alpha_\beta : E_\alpha \longrightarrow E_\beta\right)_{\alpha\leq\beta<\lambda}$ and any morphism $u : W \longrightarrow E_\lambda$ there exists a $\gamma < \lambda$ and a morphism $u' : W \longrightarrow E_\gamma$ such that $u = e^{\gamma+1}_\lambda \cdot e^\gamma_{\gamma+1} \cdot u'$.*

We can provide now the main result of this section.

Theorem 4.19. *Let $\mathscr{C}$ be an* **Ord**-*category and $\mathcal{H}$ a set of maps. Moreover, let $\mathscr{C}$ admit transfinite lax compositions of lax $\mathcal{H}$-cell ζ-complexes for every ordinal ζ, cocomma objects along morphisms in ${}^{\pitchfork}\left(\mathcal{H}^{\pitchfork}\right)$ and 2-dimensional coproducts of elements in $\mathcal{H}$. If the domains of morphisms in $\mathcal{H}$ are all laxly small relative to $\mathcal{H}$, then every morphism admits an $\left({}^{\pitchfork}\left(\mathcal{H}^{\pitchfork}\right); \mathcal{H}^{\pitchfork}\right)$-factorisation.*

Proof. Since $\mathcal{H}$ is a set and domains of morphisms in $\mathcal{H}$ are laxly small relative to $\mathcal{H}$, we have that there exists λ such that each domain H is laxly small relative to $\mathcal{H}$ with respect to some $\kappa \leq \lambda$.

We consider $f : A \longrightarrow B$. We construct a lax $\mathcal{H}$-cell λ-complex $\left(e^{\alpha}_{\beta} : E_\alpha \longrightarrow E_\beta\right)_{\alpha \leq \beta < \lambda}$ and a cocone $(p^{\alpha} : E_\alpha \longrightarrow B)_{\alpha<\lambda}$ for such diagram, i.e. such that

$$\begin{tikzcd}
E_0 \arrow[r, "e^0_\alpha"] \arrow[dr, "p^0"'] & E_\alpha \arrow[d, "p^\alpha"] \\
& B
\end{tikzcd}$$

for any $\alpha < \lambda$.

This lax $\mathcal{H}$-cell λ-complex is inductively defined as follows. We set $E_0 = A$ and $p^0 = f$.

Let γ be an ordinal. We consider the lax squares of the form

$$\begin{tikzcd}
A_j \arrow[r, "u_j"] \arrow[d, "h_j"'] \arrow[dr, phantom, "\Rightarrow"] & E_\gamma \arrow[d, "p^\gamma"] \\
B_j \arrow[r, "v_j"'] & B,
\end{tikzcd} \tag{4.x}$$

such that $h_j \in \mathcal{H}$ and we index these lax squares by $j \in I$. We build through the 2-dimensional coproducts the morphism

$$\hat{h} = \coprod_{i\in I} h_i : \coprod_{i\in I} A_i \longrightarrow \coprod_{i\in I} B_i. \tag{4.xi}$$

as in Definition 4.2. We remark that $\hat{h} \in {}^{\pitchfork}\left(\mathcal{H}^{\pitchfork}\right)$ by Proposition 4.10, since $h_i \in \mathcal{H} \subseteq {}^{\pitchfork}\left(\mathcal{H}^{\pitchfork}\right)$ for every $i \in I$. Then we can build the following cocomma object

$$\begin{tikzcd}
\coprod_{i\in I} A_i \arrow[r, "{[u_i]_{i\in I}}"] \arrow[d, "\hat{h}"'] \arrow[dr, phantom, "\Rightarrow"] & E_\gamma \arrow[d, "e^\gamma_{\gamma+1}"] \arrow[ddr, bend left, "p^\gamma"] & \\
\coprod_{i\in I} B_i \arrow[r, "q_\alpha"'] \arrow[drr, bend right, "{[v_i]_{i\in I}}"'] & E_{\gamma+1} \arrow[dr, dashed, "p^{\gamma+1}"] & \\
& & B.
\end{tikzcd} \tag{4.xii}$$

In particular $e^{\gamma}_{\gamma+1}$ is a morphism obtained from $\hat{h} \in {}^{\pitchfork}\left(\mathcal{H}^{\pitchfork}\right)$ via a cocomma object, therefore $e^{\gamma}_{\gamma+1} \in {}^{\pitchfork}\left(\mathcal{H}^{\pitchfork}\right)$ by Proposition 4.9. Then we define $e^{\alpha}_{\gamma+1} = e^{\gamma}_{\gamma+1} \cdot e^{\alpha}_{\gamma}$, which yields that $e^{0}_{\gamma+1} = e^{\gamma}_{\gamma+1} \cdot e^{0}_{\gamma}$ is still a morphism in ${}^{\pitchfork}\left(\mathcal{H}^{\pitchfork}\right)$, since it is closed under composition. Moreover, we have that $p^{\gamma} = p^{\gamma+1} \cdot e^{\gamma}_{\gamma+1}$, hence we conclude that

$$\begin{cases} p^0 = p^{\gamma} \cdot e^0_{\gamma} = p^{\gamma+1} \cdot e^{\gamma}_{\gamma+1} \cdot e^0_{\gamma} = p^{\gamma+1} \cdot e^0_{\gamma+1}; \\ e^0_{\gamma+1} = e^{\gamma}_{\gamma+1} \cdot e^0_{\gamma} = e^{\gamma}_{\gamma+1} \cdot e^{\alpha}_{\gamma} \cdot e^0_{\alpha} = e^{\alpha}_{\gamma+1} \cdot e^0_{\alpha}. \end{cases} \tag{4.xiii}$$

Hence $\left(e^{\alpha}_{\beta} : E_\alpha \longrightarrow E_\beta\right)_{\alpha \leq \beta < \gamma+1}$ is a lax $\mathcal{H}$-cell $(\gamma+1)$-complex and $(p^{\alpha} : E_\alpha \longrightarrow B)_{\alpha < \gamma+1}$ is a cocone for the diagram $\left(e^{0}_{\alpha}\right)_{\alpha<\gamma+1}$.

Now for each limit ordinal $\kappa \leq \lambda$ we have a lax $\mathcal{H}$-cell κ-complex $\left(e^{\alpha}_{\beta} : E_\alpha \longrightarrow E_\beta\right)_{\alpha \leq \beta < \kappa}$ and a cocone $(p^{\alpha} : E_\alpha \longrightarrow B)_{\alpha<\kappa}$ such that $p^0 = p^{\alpha} \cdot e^{0}_{\alpha}$ by inductive hypothesis. Hence we may write the following diagram

$$
\begin{tikzcd}
E_0 \arrow[rr, "e^0_\alpha"] \arrow[dr, "e^0_\kappa"'] \arrow[ddr, bend right, "p^0"'] & & E_\alpha \arrow[dl, "e^\alpha_\kappa"] \arrow[ddl, bend left, "p^\alpha"] \\
& E_\kappa \arrow[d, "p^\kappa"] \arrow[u, phantom, "\leq"] & \\
& B, &
\end{tikzcd}
\tag{4.xiv}
$$

where $(e^{\alpha}_{\kappa})_{\alpha\leq\kappa}$ is a lax colimit cocone for the diagram $\left(e^{0}_{\alpha}\right)_{\alpha<\kappa}$ and p_κ is the universal morphism for the cocone $(p^{\alpha})_{\alpha<\kappa}$. In particular we have for any $\alpha \leq \kappa$

$$
\begin{cases}
p^{\alpha} = p^{\kappa} \cdot e^{\alpha}_{\kappa} \\
e^{0}_{\kappa} \leq e^{\alpha}_{\kappa} \cdot e^{0}_{\alpha}.
\end{cases}
$$

Furthermore, e^{0}_{κ} is in $\pitchfork\left(\mathcal{H}^{\pitchfork}\right)$ since it is a transfinite lax composition and e^{0}_{α} belongs to $\pitchfork\left(\mathcal{H}^{\pitchfork}\right)$ for each $\alpha < \kappa$.

Therefore, we have that $\left(e^{\alpha}_{\beta} : E_\alpha \longrightarrow E_\beta\right)_{\alpha\leq\beta\leq\kappa}$ is a lax $\mathcal{H}$-cell $\overline{\kappa}$-complex and $(p^{\alpha} : E_\alpha \longrightarrow B)_{\alpha\leq\kappa}$ is a cocone for the diagram $\left(e^{0}_{\alpha}\right)_{\alpha\leq\kappa}$.

Then we take the factorisation

$$
\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e^0_\lambda"'] & & B, \\
& E_\lambda. \arrow[ur, "p^\lambda"'] &
\end{tikzcd}
$$

We remark that e^{0}_{λ} is a transfinite lax composition of morphisms in $\pitchfork\left(\mathcal{H}^{\pitchfork}\right)$, hence it belongs to $\pitchfork\left(\mathcal{H}^{\pitchfork}\right)$.

We need to prove that $p^{\lambda} \in \mathcal{H}^{\pitchfork}$. Let us consider a lax square $(u,v) : h \longrightarrow p^{\lambda}$, with $h \in \mathcal{H}$. Thus by our smallness hypothesis, we have that $u = e^{\gamma+1}_{\lambda} \cdot e^{\gamma}_{\gamma+1} \cdot u'$ for some $u' : H \longrightarrow E_\gamma$ and for some $\gamma < \lambda$. We write explicitly the diagram

$$
\begin{tikzcd}
H \arrow[r, "u'"] \arrow[rrr, bend left, "u"] \arrow[d, "h"'] \arrow[dr, phantom, "\geq"] & E_\gamma \arrow[r, "e^\gamma_{\gamma+1}"] \arrow[d, "p^\gamma"] & E_{\gamma+1} \arrow[r, "e^{\gamma+1}_\lambda"] \arrow[dl, "p^{\gamma+1}"] & E_f \arrow[dll, "p^\lambda"] \\
H' \arrow[r, "v"'] \arrow[urr, dotted] & B. & &
\end{tikzcd}
\tag{4.xv}
$$

We remark that $p^{\gamma+1} = p^{\lambda} \cdot e_{\lambda}^{\gamma+1}$ as shown in (4.xiv) and $p^{\gamma} = p^{\gamma+1} \cdot e_{\gamma}^{\gamma+1}$ by construction, as presented in (4.xii). Then we obtain the following 2-cell

$$p^{\gamma} \cdot u' = p^{\gamma+1} \cdot e_{\gamma+1}^{\gamma} \cdot u' = p^{\lambda} \cdot e_{\lambda}^{\gamma+1} \cdot e_{\gamma+1}^{\gamma} \cdot u' = p^{\lambda} \cdot u \leq v \cdot h.$$

In particular this yields that there exists a $j \in I$ such that $(u', v) : h \longrightarrow p^{\gamma}$ is one of the lax squares described in (4.x) and it is part of the construction of $\hat{h}$ in (4.xi) used to build $p^{\gamma+1}$. Thus, recalling the diagram in (4.xii), we have the following diagram

$$\begin{tikzcd}
H \arrow[r, "\sigma_h"] \arrow[d, "h"'] \arrow[rr, bend left, "u'"] & \coprod_{i \in I} A_i \arrow[r, "{[u_i]_{i \in I}}"] \arrow[d, "\hat{h}"] \arrow[dr, phantom, "\leq"] & E_{\gamma} \arrow[d, "e_{\gamma+1}^{\gamma}"'] \arrow[dr, "p^{\gamma}"] & \\
H' \arrow[r, "\sigma'_h"] \arrow[rrr, bend right, "v"'] & \coprod_{i \in I} B_i \arrow[r, "q_{\gamma}"] \arrow[rr, bend right, "{[v_i]_{i \in I}}"] & E_{\gamma+1} \arrow[r, "p^{\gamma+1}"] & B.
\end{tikzcd}$$

Hence one possible dotted morphism in diagram (4.xv) is given by $q_{\gamma} \cdot \sigma'_h$. Then we prove that

$$H' \xrightarrow{\sigma'_h} \coprod_{i \in I} B_i \xrightarrow{q_{\gamma}} E_{\gamma+1} \xrightarrow{e_{\lambda}^{\gamma+1}} E_f$$

is a diagonal lifting for (u, v). In fact we have that

$$\begin{cases} p^{\lambda} \cdot e_{\lambda}^{\gamma+1} \cdot q_{\gamma} \cdot \sigma'_h = p^{\gamma+1} \cdot q_{\gamma} \cdot \sigma'_h = [v_i]_{i \in I} \cdot \sigma'_h = v \\ u = e_{\lambda}^{\gamma+1} \cdot e_{\gamma+1}^{\gamma} \cdot u' = e_{\lambda}^{\gamma+1} \cdot e_{\gamma+1}^{\gamma} \cdot [u_i]_{i \in I} \cdot \sigma_h \leq e_{\lambda}^{\gamma+1} \cdot q_{\gamma} \cdot \sigma'_h \cdot h. \end{cases}$$

Hence $p^{\lambda} \in \mathcal{H}^{\pitchfork}$. ☐

This yields the following corollary

Corollary 4.20. *Let $\mathscr{C}$ be an* **Ord***-category and $\mathcal{H}$ a set of maps. Moreover, let $\mathscr{C}$ admit transfinite lax compositions of lax $\mathcal{H}$-cell λ-complexes for any ordinal λ, cocomma objects along morphisms in ${}^{\pitchfork}(\mathcal{H}^{\pitchfork})$ and 2-dimensional coproducts of elements in $\mathcal{H}$. If the domains of morphisms in $\mathcal{H}$ are all laxly small relative to $\mathcal{H}$, then $\left({}^{\pitchfork}(\mathcal{H}^{\pitchfork}); \mathcal{H}^{\pitchfork}\right)$ is a* $\mathrm{WFS}_{\mathrm{lax}}$.

Example 4.21. *We consider the category* **Ord** *and the set $\mathcal{O} = \{\iota_1 : \varnothing \longrightarrow 1\}$. We consider a lax square $(u, v) : \iota_1 \longrightarrow f$,*

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d, "\iota_1"'] \arrow[dr, phantom, "\leq", near start] & X \arrow[d, "f"] \\
1 \arrow[r, "v"'] \arrow[ur, dotted, "d"] \arrow[ur, phantom, "\leq", near end] & Y;
\end{tikzcd}$$

the diagonal d exists if and only if for each $y \in Y$ there exists $x \in X$ such that $f(x) \leq y$. This is evident remarking that the upper triangle is always commutative. Thus we can define

$$\mathcal{O}^{\pitchfork} = \{f : X \longrightarrow Y | \forall y \in Y \,(\exists x \in X\,(f(x) \leq y))\}.$$

We remark that the following map

$$\begin{aligned} f : \{a,b\} &\longrightarrow \{0 \leq 1\} \\ a &\longmapsto 0 \\ b &\longmapsto 1 \end{aligned}$$

belongs to $\mathcal{O}^{\pitchfork}$ and it is not a left adjoint morphism, therefore we conclude that $\mathsf{LA} \subsetneq \mathcal{O}^{\pitchfork} \subsetneq \mathsf{All}$. Hence $\left({}^{\pitchfork}\left(\mathcal{O}^{\pitchfork}\right), \mathcal{O}^{\pitchfork}\right)$ is a non-trivial lax weak prefactorisation system. Furthermore, since the domain $\varnothing$ of ι_1 is trivially laxly small relative to $\mathcal{H}$, by the previous corollary $\left({}^{\pitchfork}\left(\mathcal{O}^{\pitchfork}\right), \mathcal{O}^{\pitchfork}\right)$ is a $\mathrm{WFS}_{\mathrm{lax}}$.

4.3 Lax functorial factorisations

A natural step forward is to study factorisation systems for lax arrow categories that are functorial. To do so, we consider the composition functor applied to lax arrow categories

$$\mathscr{C}^2_{\mathrm{lax}} \times_{\mathscr{C}} \mathscr{C}^2_{\mathrm{lax}} \overset{\xrightarrow{\pi_1}}{\underset{\xrightarrow[\pi_2]{}}{\xrightarrow{(-\cdot-)}}} \mathscr{C}^2_{\mathrm{lax}}. \tag{4.xvi}$$

We point out that objects in $\mathscr{C}^2_{\mathrm{lax}} \times_{\mathscr{C}} \mathscr{C}^2_{\mathrm{lax}}$ are pairs of composable morphisms and the arrows are triples of morphisms as

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\urcorner\!\!/"] & A' \arrow[d, "f'"] \\ B \arrow[r, "b"] \arrow[d, "g"'] \arrow[dr, phantom, "\urcorner\!\!/"] & B' \arrow[d, "g'"] \\ C \arrow[r, "c"] & C'. \end{tikzcd}$$

Then the following definition becomes a natural translation from the ordinary factorisation systems.

Definition 4.22. *A **lax functorial factorisation system** is an* **Ord***-functor $F : \mathscr{C}^2_{\mathrm{lax}} \longrightarrow \mathscr{C}^2_{\mathrm{lax}} \times_{\mathscr{C}} \mathscr{C}^2_{\mathrm{lax}}$ such that $(-\cdot-)F = \mathrm{Id}_{\mathscr{C}^2_{\mathrm{lax}}}$.*

A lax functorial factorisation system is then determined by a section of the composition functor applied to $\mathscr{C}^2_{\mathrm{lax}}$. We can determine its components through the following compositions

$$L = \pi_1 \cdot F : \mathscr{C}^2_{\mathrm{lax}} \longrightarrow \mathscr{C}^2_{\mathrm{lax}} \qquad R = \pi_2 \cdot F : \mathscr{C}^2_{\mathrm{lax}} \longrightarrow \mathscr{C}^2_{\mathrm{lax}} \qquad K = \mathrm{cod} \cdot \pi_1 \cdot F = \mathrm{dom} \cdot \pi_2 \cdot F : \mathscr{C}^2_{\mathrm{lax}} \longrightarrow \mathscr{C}. \tag{4.xvii}$$

A lax functorial factorisation system induces also the natural transformations $\eta : \mathrm{Id} \Rightarrow R$ and $\varepsilon : L \Rightarrow \mathrm{Id}$. Unlike ordinary factorisation systems, these transformations are not strict in general, but only oplax.

In fact, considering η, for any lax square $(u,v) : f \longrightarrow g$ we have that

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq"] & C \arrow[r, "Lg"] \arrow[d, "g"] \arrow[dr, phantom, "\eta_g"] & Kg \arrow[d, "Rg"] \\
B \arrow[r, "v"'] & D \arrow[r, "\mathrm{id}_D"'] & D
\end{tikzcd}
\quad \leq \quad
\begin{tikzcd}
A \arrow[r, "Lf"] \arrow[d, "f"'] \arrow[dr, phantom, "\eta_f"] & Kf \arrow[r, "K(u,v)"] \arrow[d, "Rf"] \arrow[dr, phantom, "\leq"] & Kg \arrow[d, "Rg"] \\
B \arrow[r, "\mathrm{id}_B"'] & B \arrow[r, "v"'] & D,
\end{tikzcd}
$$

since $Lg \cdot u \leq K(u,v) \cdot Lf$ by definition of lax functorial factorisation. This amounts to having that $\eta_g \cdot (u,v) \leq R(u,v) \cdot \eta_f$. By analogous arguments, one can observe that ε is an oplax natural transformation as well.

4.3.1 Lax functorial weak factorisation systems

Our goal now is to interlink the two concepts as we have already seen in 2.3. We say that an ordinary weak factorisation system $(\mathcal{L},\mathcal{R})$ underlies a lax functorial factorisation system (F,L,R,K) if for every morphism f, $Rf \cdot Lf$ is also an $(\mathcal{L},\mathcal{R})$-factorisation; then (F,L,R,K) is said to be the **functorial realisation** of $(\mathcal{L},\mathcal{R})$.

We already know that, if an ordinary WFS $(\mathcal{L},\mathcal{R})$ admits a functorial realisation (F,L,R,K), then $(\mathcal{L},\mathcal{R}) = (L\text{-Coalg}, R\text{-Alg})$. This means that $\mathcal{L}$ contains those morphisms whose right component is a split epimorphism and $\mathcal{R}$ those morphisms whose left component is a split monomorphism, as we have presented in 2.25.

Following this idea we aim to investigate the conditions under which a lax functorial factorisation system is the functorial realisation of a WFS_{lax} and to give a description of the latter.

We fix a lax functorial factorisation system with components (F,L,R,K). We consider the two classes

$$\mathcal{L}_F = \{f | f \pitchfork Rf\} \qquad\qquad \mathcal{R}_F = \{f | Lf \pitchfork f\}. \tag{4.xviii}$$

Similarly to ordinary functorial WFS, we consider a lax functorial factorisation system such that for every f, $Lf \in \mathcal{L}_F$ and $Rf \in \mathcal{R}_F$. More precisely, we are considering lax factorisation systems such that $Lf \pitchfork RLf$ and $LRf \pitchfork Rf$ for every morphism f; we will call such lax functorial factorisation systems *predistributive*. The reason for this name is that the assumption amounts to a certain distributivity of the lax functorial factorisation system as depicted in (4.xxvi).

Proposition 4.23. *Let (F,L,R,K) be a lax functorial factorisation system and f any morphism. If $Rf \in \mathcal{R}_F$, then $f \in \mathcal{L}_F$ if and only if there exists a lax diagonal lifting ρ_f in the square η_f. If $Lf \in \mathcal{L}_F$, then $f \in \mathcal{R}_F$ if and only if there exists a lax diagonal lifting λ_f in the square ε_f. For any f, we will* *denote such lax liftings by*

$$
\begin{tikzcd}
A \arrow[r, "Lf"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq", near start] & Kf \arrow[d, "Rf"] \\
B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "\rho_f"] \arrow[ur, phantom, "\leq", near end, shift right=3] & B,
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "Lf"'] \arrow[dr, phantom, "\leq", near start] & A \arrow[d, "f"] \\
Kf \arrow[r, "Rf"'] \arrow[ur, "\lambda_f"] \arrow[ur, phantom, "\leq", near end, shift right=3] & B.
\end{tikzcd}
\tag{4.xix}
$$

Proof. We prove only the first statement, since the second one follows by duality.

One direction is trivial as the existence of such a ρ_f is a direct consequence of $f \pitchfork Rf$.

For the non-trivial implication, we need to prove that $f \pitchfork Rf$. We consider a lax square

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq"] & Kf \arrow[d, "Rf"] \\
B \arrow[r, "v"'] & B
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "Lf"'] \arrow[dr, phantom, "\leq"] & Kf \arrow[d, "LRf"'] \\
Kf \arrow[r, "K(u,v)"] \arrow[d, "Rf"] \arrow[dr, phantom, "\leq"] & KRf \arrow[d, "RRf"] \arrow[u, bend right=60, "\lambda_{Rf}"'] \\
B \arrow[r, "v"'] \arrow[u, bend left=60, "\rho_f"] & B,
\end{tikzcd}
\tag{4.xx}
$$

where ρ_f is a diagonal morphism of η_f existing by assumption and λ_{Rf} is a diagonal morphism of ε_{Rf} existing since $LRf \pitchfork Rf$.

We consider $\delta = \lambda_{Rf} \cdot K(u,v) \cdot \rho_f$. Bearing in mind the definitions of ρ_f and λ_{Rf} as lax diagonal morphisms for η_f and ε_{Rf} as shown in (4.xix), we have that

$$u \leq \lambda_{Rf} \cdot LRf \cdot u \leq \lambda_{Rf} \cdot K(u,v) \cdot Lf \leq \lambda_{Rf} \cdot K(u,v) \cdot \rho_f \cdot Rf \cdot Lf = \delta \cdot f \tag{4.xxi}$$

and

$$Rf \cdot \delta = RRf \cdot LRf \cdot \lambda_{Rf} \cdot K(u,v) \cdot \rho_f \leq RRf \cdot K(u,v) \cdot \rho_f \leq v \cdot Rf \cdot \rho_f \leq v. \tag{4.xxii}$$

This implies that δ is the diagonal morphism sought and $f \pitchfork Rf$. $\square$

Corollary 4.24. *Let (F,L,R,K) be a predistributive lax functorial factorisation system. Then*

$$
\begin{aligned}
\mathcal{L}_F &= \{f \mid \eta_f \text{ has a lax diagonal morphism}\} \\
\mathcal{R}_F &= \{f \mid \varepsilon_f \text{ has a lax diagonal morphism}\}.
\end{aligned}
\tag{4.xxiii}
$$

We observe that, given a morphism f that lies in $\mathcal{L}_F \cap \mathcal{R}_F$, then the composition $\lambda_f \cdot \rho_f$ is a right adjoint to f.

Remark 4.25. *We would like to remark some interesting consequences of Corollary 4.24. Let us consider a lax predistributive* $\mathrm{FFS}_{\mathrm{lax}}$ *(F,L,R,K). Let $f \in \mathcal{R}_F$. Then there exists the lax diagonal morphism λ_f for the square ε_f, as depicted in diagram* (4.xix)*. Thus we can write the following* *diagram*

$$
\begin{tikzcd}
A \arrow[r, "Lf"] \arrow[d, "f"'] \arrow[rr, bend left=40, "\mathrm{id}_A"] \arrow[r, phantom, bend left=20, shift left=3, "\geq"] & Kf \arrow[r, "\lambda_f"] \arrow[d, "Rf"'] \arrow[dr, phantom, "\leq"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{id}_B"'] \arrow[rr, bend right=40, "\mathrm{id}_B"'] \arrow[ur, phantom, "\eta_f"] & B \arrow[r, "\mathrm{id}_B"'] & B.
\end{tikzcd}
\tag{4.xxiv}
$$

This yields that $\mathrm{id}_f \leq \left(\lambda_f, \mathrm{id}_B\right) \cdot \eta_f$.

We consider the pair (R, η) *and call it a* lax pointed endofunctor*, generalising Definition 2.24. Then a lax R-algebra is a pair* (f, α) *such that*

$$
\begin{tikzcd}
f \arrow[r, "\eta_f"] \arrow[dr, "\mathrm{id}_f"'] \arrow[dr, phantom, shift left=3, "\leq"] & Rf \arrow[d, "\alpha"] \\
& f.
\end{tikzcd}
$$

The motivation for this 2-cell and its direction is that we are interested to approach the algebras for lax monads, as we will see in Section 4.4. Then diagram (4.xxiv) *yields that* $\left(f, \left(\lambda_f, \mathrm{id}_B\right)\right)$ *is a lax R-algebra. Hence, if* $f \in \mathcal{R}_F$*, then* f *is a lax R-algebra, furthermore* $\mathcal{R}_F \subseteq R\text{-}\mathrm{Alg}_{\mathrm{lax}}$.

On the other hand, let us consider a lax R-algebra (f, α)*. Then, we can write the following diagram*

$$
\begin{tikzcd}
A \arrow[r, "Lf"] \arrow[d, "f"'] \arrow[rr, bend left=40, "\mathrm{id}_A"] \arrow[r, phantom, bend left=20, shift left=3, "\geq"] & Kf \arrow[r, "\alpha_0"] \arrow[d, "Rf"'] \arrow[dr, phantom, "\leq"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{id}_B"'] \arrow[rr, bend right=40, "\mathrm{id}_B"'] \arrow[ur, phantom, "\eta_f"] & B \arrow[r, "\alpha_1"'] & B.
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{cases}
f \cdot \alpha_0 \leq \alpha_1 \cdot Rf \\
\mathrm{id}_B \leq \alpha_1 \\
\mathrm{id}_A \leq \alpha_0 \cdot Lf.
\end{cases}
\tag{4.xxv}
$$

If we attempt to analyse whether α_0 *is a lax diagonal lifting for* ε_f*, we only get*

$$
\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dd, "Lf"'] \arrow[ddrr, phantom, shift right=6, "\geq"] & & A \arrow[dd, "f"] \arrow[dl, phantom, shift left=3, "\geq"] \\
& B \arrow[dr, "\alpha_1"] \arrow[d, phantom, "\leq"] & \\
Kf \arrow[rr, "Rf"'] \arrow[uurr, "\alpha_0"] \arrow[ur, "Rf"] & \; & B.
\end{tikzcd}
$$

So we observe that in general not all lax R-algebras belong to $\mathcal{R}_F$. Dual arguments can be carried out to deduce that $\mathcal{L}_F \subseteq L\text{-}\mathrm{Coalg}_{\mathrm{lax}}$.

According to these remarks, the choice of the name predistributive points to the existence, for any morphism f, of the following diagram

$$\begin{tikzcd} Kf \arrow[r, "\rho_{Lf}"] \arrow[d, "LRf"'] \arrow[dr, equal] \arrow[dr, phantom, shift right=3ex, "\leq"] \arrow[dr, phantom, shift left=3ex, "\leq"] & KLf \arrow[d, "RLf"] \\ KRf \arrow[r, "\lambda_{Rf}"'] & Kf \end{tikzcd} \tag{4.xxvi}$$

and it coincides with the assumption that for every morphism f, $Rf \in \mathcal{R}_F$ and $Lf \in \mathcal{L}_F$.

This diagram resembles the distributivity transformation described in the next section, even if it carries less structure. In fact, it is not in general a natural transformation and it does not satisfy any distributivity law as we have described in Definition 2.27 and Remark 2.28.

Theorem 4.26. *Let (F,L,R,K) be a predistributive* $\mathrm{FFS}_{\mathrm{lax}}$. *Then $(\mathcal{L}_F,\mathcal{R}_F)$ is a* $\mathrm{WFS}_{\mathrm{lax}}$. *Moreover, if a* $\mathrm{WFS}_{\mathrm{lax}}$ *$(\mathcal{L},\mathcal{R})$ admits a lax functorial realisation (F,L,R,K), then $(\mathcal{L},\mathcal{R}) = (\mathcal{L}_F,\mathcal{R}_F)$.*

Proof. We start by proving that $(\mathcal{L}_F,\mathcal{R}_F)$ is a $\mathrm{WFS}_{\mathrm{lax}}$. First we show that $\mathcal{L}_F \pitchfork \mathcal{R}_F$. Let $f \in \mathcal{L}_F$ and $g \in \mathcal{R}_F$. Thus, we have that $f \pitchfork Rf$ and $Lg \pitchfork g$, by the existence of the two morphisms ρ_f and λ_g. Given a lax square $(u,v) : f \longrightarrow g$, we can factorise it as follows

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\swarrow"] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd} \qquad \longmapsto \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, "Lf"'] \arrow[dr, phantom, "\swarrow"] & C \arrow[d, "Lg"] \\ Kf \arrow[r, "K(u,v)"] \arrow[d, "Rf"'] \arrow[dr, phantom, "\swarrow"] & Kg \arrow[u, bend right=60, "\lambda_g"'] \arrow[d, "Rg"] \\ B \arrow[r, "v"'] \arrow[u, bend left=60, "\rho_f"] & D. \end{tikzcd} \tag{4.xxvii}$$

Then, we notice that the morphism $\delta = \lambda_g \cdot K(u,v) \cdot \rho_f$ constitutes a lax diagonal morphism for the lax square taken into account. Therefore we have that $\mathcal{L}_F \pitchfork \mathcal{R}_F$.

Moreover, for any $f \pitchfork \mathcal{R}_F$, it follows that $f \pitchfork Rf$, since $Rf \in \mathcal{R}_F$ by lax predistributivity, which yields that $f \in \mathcal{L}_F$, namely $\mathcal{R}_F^{\pitchfork} \subseteq \mathcal{L}_F$. By an analogous argument $\mathcal{L}_F^{\pitchfork} \subseteq \mathcal{R}_F$ holds.

We focus now on the second claim. Let $(\mathcal{L},\mathcal{R})$ be a $\mathrm{WFS}_{\mathrm{lax}}$ admitting a lax functorial realisation (F,L,R,K). We have have that

$$\begin{array}{ccccccc} f \in \mathcal{L} & \Leftrightarrow & f \in {}^{\pitchfork}\mathcal{R} & \Rightarrow & f \pitchfork Rf & \Leftrightarrow & f \in \mathcal{L}_F; \\ f \in \mathcal{R} & \Leftrightarrow & f \in \mathcal{L}^{\pitchfork} & \Rightarrow & Lf \pitchfork f & \Leftrightarrow & f \in \mathcal{R}_F; \end{array}$$

so we have that $\mathcal{L} \subseteq \mathcal{L}_F$ and $\mathcal{R} \subseteq \mathcal{R}_F$. Furthermore, we know that orthogonal complements are antitone with respect to inclusion by Proposition 4.5, hence these two inclusions yield $\mathcal{R}_F \subseteq \mathcal{R}$ and $\mathcal{L}_F \subseteq \mathcal{L}$ respectively. In conclusion, we have that $(\mathcal{L},\mathcal{R}) = (\mathcal{L}_F,\mathcal{R}_F)$. $\square$

The last theorem gives us a description of those $\mathrm{WFS}_{\mathrm{lax}}$ that admit a lax functorial realisation. Moreover, for any $\mathrm{FFS}_{\mathrm{lax}}$ it states that, if every L-component and every R-component bear particular lax weak orthogonality relations, then they form the unique underlying $\mathrm{WFS}_{\mathrm{lax}}$.

4.4 Lax Algebraic Factorisation Systems

In this section we present a class of functorial factorisation systems which are in general predistributive and come equipped with a richer structure close to the one of a monad. This construction mirrors in this lax context that of algebraic weak factorisation systems, which we encountered in Section 2.4.

We recall the definition of lax monad that we reprise from [Bun74]. We define here an **Ord** version of this definition and, although this work refers to lax natural transformations, we remark that it is actually the same type of transformation we call oplax according to what appears to be the most used choice in literature. The only difference is that we will use a definition involving usual functors and not lax functors, since it is the particularisation that best fits our purposes.

Definition 4.27. *For an* **Ord***-enriched category $\mathscr{C}$, a* ***lax monad*** *is a triple (T,η,μ) such that*

- $T:\mathscr{C}\longrightarrow\mathscr{C}$ is a functor;
- $\eta:\mathrm{Id}\Longrightarrow T$ is an oplax natural transformation;
- $\mu:TT\Longrightarrow T$ is an oplax natural transformation;

and such that the following lax monads laws are satisfied

$$\begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "\mathrm{id}_T"'] & TT \arrow[d, "\mu"] \arrow[dl, phantom, "\leq"] \arrow[dr, phantom, "\leq"] & T \arrow[l, "\eta_T"'] \arrow[dl, "\mathrm{id}_T"] \\ & T & \end{tikzcd} \qquad \begin{tikzcd} TTT \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] \arrow[dr, phantom, "\leq"] & TT \arrow[d, "\mu"] \\ TT \arrow[r, "\mu"'] & T. \end{tikzcd} \tag{4.xxviii}$$

We now define the factorisation systems we are interested in.

Definition 4.28. *A* ***lax algebraic weak factorisation system*** *($\mathrm{AWFS}_{\mathrm{lax}}$) is a lax functorial factorisation system (F,L,R,K) such that (R,η) is part of a lax monad (R,η,Θ), (L,ε) is part of a lax comonad (L,ε,Ω), and there is a natural transformation $\Delta=(\mathrm{cod}(\sigma),\mathrm{dom}(\pi)):LR\Longrightarrow RL$ which constitues a distributive law of the comonad over the monad in the sense that the following diagram commutes*

$$\begin{tikzcd} LRR \arrow[r, "\Delta_R"] \arrow[d, "L\Theta"'] & RLR \arrow[r, "R\Delta"] & RRL \arrow[d, "\Theta_L"] \\ LR \arrow[rr, "\Delta"] \arrow[d, "\Omega_R"'] & & RL \arrow[d, "R\Omega"] \\ LLR \arrow[r, "L\Delta"] & LRL \arrow[r, "\Delta_L"] & RLL. \end{tikzcd} \tag{4.xxix}$$

As said before, these factorisation systems constitute a subclass of lax functorial weak factorisation systems as we prove in the following proposition.

Proposition 4.29. *A* AWFS$_{\mathrm{lax}}$ (F,L,R,K) *is the functorial realisation of a* WFS$_{\mathrm{lax}}$.

Proof. We can prove the statement by showing that (F,L,R,K) is lax predistributive. Let f be any morphism. We consider the lax square given by Θ_f,

$$\begin{tikzcd}
KRf \arrow[r, "\theta_f"] \arrow[d, "RRf"'] \arrow[dr, phantom, "\leq"] & Kf \arrow[d, "Rf"] \\
B \arrow[r, "\mathrm{id}_B"'] & B.
\end{tikzcd}$$

We remark that the lax monad axioms in (4.xxviii), in particular the left diagram, yield that the codomain component of Θ is an identity since the codomain component of η is an identity. Then, by the lax monad law $\mathrm{id}_R \leq \Theta \cdot \eta_R$, we can deduce, restricting it to the domains, that $\mathrm{id}_{Kf} \leq \theta_f \cdot LRf$. Thus we have that θ_f is a lax diagonal morphism for ε_{Rf}, i.e.

$$\begin{tikzcd}
Kf \arrow[r, "\mathrm{id}_{Kf}"] \arrow[d, "LRf"'] & Kf \arrow[d, "Rf"] \\
KRf \arrow[r, "RRf"'] \arrow[ur, "\theta_f" description] & B.
\end{tikzcd}$$

The same argument on the comonad yields that ω_f, the codomain morphism of the comultiplication Ω of the comonad, is the lax diagonal morphism for η_{Lf}. Now we want to check that $Lf \pitchfork RLf$ and $LRf \pitchfork Rf$. We prove only the first one, since the arguments for the second are similar. We consider any lax square $(u,v) : Lf \longrightarrow RLf$ and its factorisation

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "LLf"'] \arrow[dr, phantom, "\leq"] & KLf \arrow[d, "LRLf"'] \\
KLf \arrow[r, "K(u,v)"] \arrow[d, "RLf"] \arrow[dr, phantom, "\leq"] & KRLf \arrow[d, "RRLf"] \arrow[u, bend right, "\theta_{Lf}"'] \\
Kf \arrow[r, "v"'] \arrow[u, bend left, "\omega_f"] & Kf.
\end{tikzcd}$$

Then $\theta_{Lf} \cdot K(u,v) \cdot \omega_f$ is a lax diagonal morphism. In fact, due to the rules of the monad, we have

$$\begin{cases} u \leq \theta_{Lf} \cdot LRLf \cdot u \leq \theta_{Lf} \cdot K(u,v) \cdot LLf \leq \theta_{Lf} \cdot K(u,v) \cdot \omega_f \cdot RLf \cdot LLf \\ RRLf \cdot LRLf \cdot \theta_{Lf} \cdot K(u,v) \cdot \omega_f \leq RRLf \cdot K(u,v) \cdot \omega_f \leq v \cdot RLf \cdot \omega_f \leq v. \end{cases} \tag{4.xxx}$$

This implies that, for every f, we have that $Lf \pitchfork RLf$, and similarly $LRf \pitchfork Rf$. Therefore (F, L, R, K) is lax predistributive and hence it is the functorial realisation of the $\textsc{wfs}_{\mathrm{lax}}$ $(\mathcal{L}_F, \mathcal{R}_F)$. □

We remark that in general a lax predistributive functorial factorisation system does not yield a complete distributivity law, since it is not even true that the square (4.xxvi) is a natural transformation, and moreover we do not have the existence of the 2-cells in the distributivity law (4.xxix).

4.5 Oplax morphism categories

The arguments developed in the previous sections of this chapter are presented in the context of $\mathscr{C}^2_{\mathrm{lax}}$. We remark that this constitutes a choice on the directions of 2-cells within squares and that such a choice is not unique. In fact, any **Ord**-category $\mathscr{C}$ gives rise to another category denoted by $\mathscr{C}^2_{\mathrm{oplax}}$, whose objects are arrows in $\mathscr{C}$ and whose morphisms are oplax squares $(u, v) : f \longrightarrow g$ of the type

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq"] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D. \end{tikzcd}$$

We observe that what we have described for $\mathscr{C}^2_{\mathrm{lax}}$ can be expressed in a dual fashion for $\mathscr{C}^2_{\mathrm{oplax}}$ and it yields an equally powerful dual set of results that can be used for $\mathscr{C}^2_{\mathrm{oplax}}$.

Notation 4.30. *We provide the notation that we will use to refer to opcomma and opcocomma objects;*

$$\begin{tikzcd} f \downarrow g \arrow[r, "f^{\uparrow}(g)"] \arrow[d, "g^{\rightarrow}(f)"'] \arrow[dr, phantom, "\leq"] & A \arrow[d, "f"] \\ C \arrow[r, "g"'] & B, \end{tikzcd} \qquad \begin{tikzcd} A' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\leq"] & C' \arrow[d, "g'_{\rightarrow}(f')"] \\ B' \arrow[r, "f'_{\uparrow}(g')"'] & f \uparrow g. \end{tikzcd}$$

Since it will be useful in the following chapters, we will briefly go through some of the dual results pointing out the most important differences. First of all we define the orthogonality relation that we will use in this context.

Definition 4.31. *Let $\mathscr{C}$ be an* **Ord***-enriched category. Two morphisms in $\mathscr{C}^2$ are said to be* ***oplaxly weakly orthogonal****, denoted by $f \pitchfork g$, if, for every oplax square $(u, v) : f \longrightarrow g$, there exists a morphism d such that*

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, "d"] & D \end{tikzcd} \qquad \begin{cases} d \cdot f \leq u \\ v \leq g \cdot d. \end{cases} \tag{4.xxxi}$$

We will refer to d as ***oplax diagonal morphism*** *or oplax diagonal lifting.*

Then we have the following characterisation of self orthogonal morphisms.

Proposition 4.32. *Given a morphism $f \in \mathscr{C}^2_{\text{oplax}}$, the following properties are equivalent:*

1. *$f \pitchfork f$;*
2. *f is a right adjoint morphism to some f_*;*
3. *$f \pitchfork \mathscr{C}^2_{\text{oplax}}$;*
4. *$\mathscr{C}^2_{\text{oplax}} \pitchfork f$.*

For any class of morphisms $\mathcal{H}$ we can define its oplax weak orthogonal complements as follows

$$^{\pitchfork}\mathcal{H} = \{f | f \pitchfork h \text{ for every } h \in \mathcal{H}\} \qquad \text{and} \qquad \mathcal{H}^{\pitchfork} = \{f | h \pitchfork f \text{ for every } h \in \mathcal{H}\}.$$

Then we have that oplax orthogonal complements bear the following properties.

Proposition 4.33. *Let $\mathscr{C}$ be an* **Ord***-category. The pair $^{\pitchfork}(-)$ and $(-)^{\pitchfork}$ forms a Galois connection among the classes of morphisms in $\mathscr{C}$ partially ordered by the inclusion.*

Corollary 4.34. *Let $\mathcal{H}$ be a class of morphisms in $\mathscr{C}$. Then $\mathcal{H} \subseteq {}^{\pitchfork}(\mathcal{H}^{\pitchfork})$ and $\mathcal{H} \subseteq ({}^{\pitchfork}\mathcal{H})^{\pitchfork}$. Moreover, $\mathcal{H}^{\pitchfork} = ({}^{\pitchfork}(\mathcal{H}^{\pitchfork}))^{\pitchfork}$ and ${}^{\pitchfork}\mathcal{H} = {}^{\pitchfork}\left(({}^{\pitchfork}\mathcal{H})^{\pitchfork}\right)$.*

Proposition 4.35. *Let $\mathcal{H}$ be a class of morphisms in an* **Ord***-category $\mathscr{C}$. Then the following assertions hold.*

1. *$\mathcal{H}^{\pitchfork}$ and ${}^{\pitchfork}\mathcal{H}$ contain all the right adjoint morphisms of $\mathscr{C}$;*
2. *$\mathcal{H}^{\pitchfork}$ and ${}^{\pitchfork}\mathcal{H}$ are closed under composition;*
3. *$\mathcal{H}^{\pitchfork}$ is closed under opcomma objects and ${}^{\pitchfork}\mathcal{H}$ is closed under opcocomma objects;*

We introduce the definition of oplax weak factorisation system.

Definition 4.36. *An* ***oplax weak prefactorisation system*** *is a pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ such that $\mathcal{R} = \mathcal{L}^{\pitchfork}$ and $\mathcal{L} = {}^{\pitchfork}\mathcal{R}$. Moreover, if any morphism $f \in \mathscr{C}^2_{\text{oplax}}$ has an $(\mathcal{L}, \mathcal{R})$-factorisation*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathcal{L} \ni l_f"'] & & B \\ & W_f \arrow[ur, "r_f \in \mathcal{R}"'] & \end{tikzcd} \tag{4.xxxii}$$

then $(\mathcal{L}, \mathcal{R})$ is said to be an ***oplax weak factorisation system*** *($\text{WFS}_{\text{oplax}}$).*

Proposition 4.37. *Given a class of morphisms $\mathcal{H} \subseteq \mathscr{C}^2$, then $\left({}^{\pitchfork}(\mathcal{H}^{\pitchfork}); \mathcal{H}^{\pitchfork}\right)$ and $\left({}^{\pitchfork}\mathcal{H}; ({}^{\pitchfork}\mathcal{H})^{\pitchfork}\right)$ are oplax prefactorisation systems.*

Moreover, we have again the following method to generate oplax factorisations from a class of morphisms.

Theorem 4.38. *If $\mathscr{C}$ is an* **Ord***-category with enough oplax colimits and $\mathcal{H}$ a set of maps whose domains are all oplaxly relatively small with respect to $\mathcal{H}$, then then every morphism admits a $({}^{\pitchfork}(\mathcal{H}^{\pitchfork});\mathcal{H}^{\pitchfork})$-factorisation.*

Then there exists also the composition functor in

$$\mathscr{C}^2_{\text{oplax}} \times_{\mathscr{C}} \mathscr{C}^2_{\text{oplax}} \overset{\xrightarrow{\pi_1}}{\underset{\xrightarrow{\pi_2}}{\xrightarrow{(-\cdot-)}}} \mathscr{C}^2_{\text{oplax}}, \tag{4.xxxiii}$$

and hence we can provide the following definition.

Definition 4.39. *An* ***oplax functorial factorisation system*** *is a functor $F : \mathscr{C}^2_{\text{oplax}} \longrightarrow \mathscr{C}^2_{\text{oplax}} \times_{\mathscr{C}} \mathscr{C}^2_{\text{oplax}}$ such that $(-\cdot-)F = \mathrm{Id}_{\mathscr{C}^2_{\text{oplax}}}$.*

We can determine again the components of an $\text{FFS}_{\text{oplax}}$ through the following compositions

$$L = \pi_1 \cdot F : \mathscr{C}^2_{\text{oplax}} \longrightarrow \mathscr{C}^2_{\text{oplax}} \quad R = \pi_2 \cdot F : \mathscr{C}^2_{\text{oplax}} \longrightarrow \mathscr{C}^2_{\text{oplax}} \quad K = \mathrm{cod} \cdot \pi_1 \cdot F = \mathrm{dom} \cdot \pi_2 \cdot F : \mathscr{C}^2_{\text{oplax}} \longrightarrow \mathscr{C}. \tag{4.xxxiv}$$

We remark that an $\text{FFS}_{\text{oplax}}$ induces also the natural transformations $\eta : \mathrm{Id} \Rightarrow R$ and $\varepsilon : L \Rightarrow \mathrm{Id}$. Unlike the lax case, these are not oplax transformations anymore, but lax natural transformations.

Now we consider an $\text{FFS}_{\text{oplax}}$ with components (F, L, R, K). We have again the following two classes

$$\mathcal{L}_F = \{f | f \pitchfork Rf\} \qquad\qquad \mathcal{R}_F = \{f | Lf \pitchfork f\}. \tag{4.xxxv}$$

We will consider again an oplax functorial factorisation system such that, for every f, $Lf \in \mathcal{L}_F$ and $Rf \in \mathcal{R}_F$. We will call such $\text{FFS}_{\text{oplax}}$ *predistributive*.

Lemma 4.40. *Let (F, L, R, K) be a predistributive $\text{FFS}_{\text{oplax}}$. Then*

$$\begin{aligned} \mathcal{L}_F &= \left\{f \mid \eta_f \text{ has an oplax diagonal morphism}\right\} \\ \mathcal{R}_F &= \left\{f \mid \varepsilon_f \text{ has an oplax diagonal morphism}\right\}. \end{aligned} \tag{4.xxxvi}$$

If we consider an oplax pointed endofunctor (R, η), whose definition is the dual of a lax pointed endofunctor, namely we are assuming that η is a lax natural transformation. Then, an oplax R-algebra is a pair (f, α) such that

$$\begin{tikzcd} f \arrow[r, "\eta_f"] \arrow[dr, "\mathrm{id}_f"'] & Rf \arrow[d, "\alpha"] \arrow[dl, phantom, "\Rightarrow"] \\ & f. \end{tikzcd}$$

We notice that, again, all elements in $\mathcal{R}_F$ are oplax R-algebras and all elements in $\mathcal{L}_F$ are oplax L-coalgebras. Hence we have $\mathcal{R}_F \subseteq R\text{-Alg}_{\text{oplax}}$ and $\mathcal{L}_F \subseteq L\text{-Coalg}_{\text{oplax}}$.

Then we have again the following theorem

Theorem 4.41. *Let (F, L, R, K) be a predistributive $\text{FFS}_{\text{oplax}}$. Then $(\mathcal{L}_F, \mathcal{R}_F)$ is an $\text{WFS}_{\text{oplax}}$. Moreover, if an $\text{WFS}_{\text{oplax}}$ $(\mathcal{L}, \mathcal{R})$ admits oplax functorial realisation (F, L, R, K), then $(\mathcal{L}, \mathcal{R}) = (\mathcal{L}_F, \mathcal{R}_F)$.*

We recall that oplax monads are defined by the dual of Definition 4.27. Then we may provide the following definition.

Definition 4.42. *An **oplax algebraic weak factorisation system** (*$\mathrm{AWFS}_{\mathrm{oplax}}$*) is a functorial factorisation system* (F,L,R,K) *such that* (R,η) *is part of an oplax monad* (R,η,Θ)*,* (L,ε) *is part of an oplax comonad* (L,ε,Ω)*, and there exists a distributivity law* $\Delta\colon LR \Longrightarrow RL$ *of the comonad over the monad in the sense that the following diagram commutes*

$$\begin{tikzcd}
LRR \arrow[r, "\Delta_R"] \arrow[d, "L\Theta"'] & RLR \arrow[r, "R\Delta"] & RRL \arrow[d, "\Theta_L"] \\
LR \arrow[rr, "\Delta"] \arrow[d, "\Omega_R"'] & & RL \arrow[d, "R\Omega"] \\
LLR \arrow[r, "L\Delta"] & LRL \arrow[r, "\Delta_L"] & RLL.
\end{tikzcd} \tag{4.xxxvii}$$

We also have the following dual result.

Proposition 4.43. *An* $\mathrm{AWFS}_{\mathrm{oplax}}$ (F,L,R,K) *is an oplax functorial weak factorisation system.*

Chapter 5

Lax Factorisation systems among partial maps

In this chapter we present an application of the definitions and results provided in Chapter 4. The contents of this chapter were introduced in [Lar21]. The arguments are set in the context of categories of partial maps, that we have described in Chapter 3.

In the first section we discuss the existence of a AWFS_{lax} for a general category of partial maps. The idea of this factorisation is to separate the domain and the function components of any morphism.

Thereafter we show that any stable $\text{WFS}_{\text{oplax}}$ on an **Ord**-category $\mathscr{C}$ gives rise to an $\text{WFS}_{\text{oplax}}$ on $\mathcal{P}(\mathscr{C})$, whenever it can be constructed. Moreover, if $\mathcal{P}(\mathscr{C})$ is equipped with the **Ord**-enrichment induced by the discrete one on $\mathscr{C}$, then the $\text{WFS}_{\text{oplax}}$ on $\mathcal{P}(\mathscr{C})$ inherits the properties of being functorial or algebraically weak from the base factorisation system.

Then we discuss the restriction of $\text{WFS}_{\text{oplax}}$ from a category of partial maps to its base category, and we finally show a correspondence between the two classes of $\text{WFS}_{\text{oplax}}$ for the two categories.

We conclude the chapter discussing some remarks on fibrant and cofibrant constructions of lax factorisation systems for some pointed **Ord**-categories applying these remarks to categories of partial maps.

Along the chapter we include examples of the constructions that we will present in particular for the category of **Set** and $\mathcal{P}(\mathbf{Set})$.

5.1 Lax factorisations and total maps

We consider an **Ord**-category $\mathscr{C}$ and a class of admissible subobjects $\mathcal{S}$ such that any morphism in $\mathcal{S}$ is full and upper-closed and $s', s' \cdot s \in \mathcal{S}$ implies that $s \in \mathcal{S}$. Then, by Theorem 3.6, we get that $\sqsubseteq$ induces an **Ord**-enrichment on $\mathcal{P}_{\mathcal{S}}(\mathscr{C})$.

Given any partial map f in $\mathcal{P}(\mathscr{C})$, it can be factorised as

$$
\begin{tikzcd}
D_f \arrow[d, tail, "\sigma_f"'] \arrow[dr, "\varphi_f"] & \\
A \arrow[r, "\shortmid" marking, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
D_f \arrow[d, tail, "\sigma_f"'] \arrow[dr, "\mathrm{id}_{D_f}"] & D_f \arrow[d, tail, "\mathrm{id}_{D_f}"'] \arrow[dr, "\varphi_f"] & \\
A \arrow[r, "\shortmid" marking, "Lf"'] & D_f \arrow[r, "\shortmid" marking, "Rf"'] & B,
\end{tikzcd}
\tag{5.i}
$$

as observed in [Fio95, Chapter 2].

Proposition 5.1. *Let $\mathscr{C}$ be an* **Ord***-category that allows the construction of $\mathcal{P}(\mathscr{C})$ for a class of admissible subobjects $\mathcal{S}$ such that any morphism in $\mathcal{S}$ is full and upper-closed and $s', s' \cdot s \in \mathcal{S}$ implies that $s \in \mathcal{S}$. Then the factorisation in* (5.i) *constitutes a* $\mathrm{FFS}_{\mathrm{lax}}$

$$F : \mathcal{P}(\mathscr{C})^2_{\mathrm{lax}} \longrightarrow \mathcal{P}(\mathscr{C})^2_{\mathrm{lax}} \times_{\mathcal{P}(\mathscr{C})} \mathcal{P}(\mathscr{C})^2_{\mathrm{lax}}.$$

Proof. Our goal now will be to prove that such factorisation is functorial in the sense of Definition 4.22. Therefore we aim to prove that Lf and Rf are indeed functors. We consider a lax square $(u,v) : f \longrightarrow g$ and its factorisation

$$
\begin{tikzcd}
A \arrow[r, "\shortmid" marking, "u"] \arrow[d, "\shortmid" marking, "f"'] \arrow[dr, phantom, "\leq"] & C \arrow[d, "\shortmid" marking, "g"] \\
B \arrow[r, "\shortmid" marking, "v"'] & D
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "Lf"'] \arrow[dr, phantom, "\boxed{1}"] & C \arrow[d, "Lg"] \\
D_f \arrow[r, dashed, "K(u,v)"] \arrow[d, "Rf"'] \arrow[dr, phantom, "\boxed{2}"] & D_g \arrow[d, "Rg"] \\
B \arrow[r, "v"] & D.
\end{tikzcd}
\tag{5.ii}
$$

In particular $K(u,v)$ is defined as

$$
\begin{tikzcd}
D_{g\cdot u} \arrow[d, tail, "s"'] \arrow[ddr, "\sigma_g{}^*(\varphi_u)"] & \\
D_{v\cdot f} \arrow[d, tail, "\varphi_f{}^*(\sigma_v)"'] & \\
D_f \arrow[r, "\shortmid" marking] & D_g,
\end{tikzcd}
\tag{5.iii}
$$

where s is the morphism in $\mathcal{S}$ yielded by the 2-cell $g \cdot u \leq v \cdot f$. Moreover, $s \in \mathcal{S}$ makes the following diagram commute

$$
\begin{tikzcd}
D_{g\cdot u} \arrow[rr, tail, "s"] \arrow[d, tail, "\varphi_u{}^*(\sigma_g)"'] & & D_{v\cdot f} \arrow[d, tail, "\varphi_f{}^*(\sigma_v)"] \\
D_u \arrow[dr, tail, "\sigma_u"'] & & D_f \arrow[dl, tail, "\sigma_f"] \\
& A. &
\end{tikzcd}
$$

Consequently, we have that the compositions

$$
\begin{tikzcd}
D_{g\cdot u} \arrow[d, tail, "s"'] \arrow[ddr, "\mathrm{id}_{D_{g\cdot u}}"] \\
D_{v\cdot f} \arrow[d, tail, "\varphi_f{}^*(\sigma_v)"'] \\
D_f \arrow[dd, tail, "\sigma_f"'] \arrow[ddr, "\mathrm{id}_{D_f}"] & D_{g\cdot u} \arrow[d, tail, "s"] \arrow[ddr, "\sigma_g{}^*(\varphi_u)"] \\
& D_{v\cdot f} \arrow[d, tail, "\varphi_f{}^*(\sigma_g)"] \\
A \arrow[r, "Lf"'] & D_f \arrow[r, "K(u,v)"'] & D_g
\end{tikzcd}
\qquad
\begin{tikzcd}
D_{g\cdot u} \arrow[d, tail, "\varphi_u{}^*(\sigma_g)"'] \arrow[dr, "\sigma_g{}^*(\varphi_u)"] \\
D_u \arrow[d, tail, "\sigma_u"'] \arrow[dr, "\varphi_u"] & D_g \arrow[d, tail, "\sigma_g"] \arrow[dr, "\mathrm{id}_{D_g}"] \\
A \arrow[r, "u"'] & C \arrow[r, "Lg"'] & D_g
\end{tikzcd}
$$

are equal, thus the upper square $\boxed{1}$ in the second diagram of (5.ii) is actually commutative. Moreover, we have that the morphism s yields the 2-cell $Rg \cdot K(u,v) \leq v \cdot Rf$ and this is shown by writing explicitly the diagram

$$
\begin{tikzcd}
D_{g\cdot u} \arrow[rrr, tail, "s"] \arrow[d, tail, "\mathrm{id}_{D_{g\cdot u}}"'] \arrow[dr, "\sigma_g{}^*(\varphi_u)"] & & & D_{v\cdot f} \arrow[d, tail, "\varphi_f{}^*(\sigma_v)"'] \arrow[dr, "\sigma_v{}^*(\varphi_f)"] \\
D_{g\cdot u} \arrow[d, tail, "\varphi_f{}^*(\sigma_v)\cdot s"'] \arrow[dr, "\sigma_g{}^*(\varphi_u)"] & D_g \arrow[d, tail, "\mathrm{id}_{D_g}"] \arrow[dr, "\varphi_g"] & & D_f \arrow[d, tail, "\mathrm{id}_{D_f}"'] \arrow[dr, "\varphi_f"] & D_v \arrow[d, tail, "\sigma_v"] \arrow[dr, "\varphi_v"] \\
D_f \arrow[r, "K(u,v)"'] & D_g \arrow[r, "Rg"'] & C & D_f \arrow[r, "Rf"'] & B \arrow[r, "v"'] & C.
\end{tikzcd}
$$

We conclude that $\boxed{2}$ is a lax square.

It remains to check that K is a functor. We consider two composable lax squares $(u,v): f \longrightarrow g$ and $(u',v'): g \longrightarrow h$. We have to prove that $K(u' \cdot u, v' \cdot v) = K(u',v') \cdot K(u,v)$. Recalling the definition of K given in (5.iii) we have the following diagrams

$$
\begin{tikzcd}
P \arrow[d, tail] \arrow[dr, "a"] \\
D_{g\cdot u} \arrow[d, tail, "s"'] \arrow[ddr, "\sigma_g{}^*(\varphi_u)"] & D_{h\cdot u'} \arrow[d, tail, "s'"'] \arrow[ddr, "\sigma_h{}^*(\varphi_{u'})"] \\
D_{v\cdot f} \arrow[d, tail, "\varphi_f{}^*(\sigma_v)"'] & D_{v'\cdot g} \arrow[d, tail, "\varphi_g{}^*(\sigma_{v'})"] \\
D_f \arrow[r, "K(u,v)"'] & D_g \arrow[r, "K(u',v')"'] & D_h
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
D_{h\cdot u'\cdot u} \arrow[d, tail, "s''"'] \arrow[dddr, "\sigma_h{}^*(\varphi_{u'\cdot u})"] \\
D_{v'\cdot v\cdot f} \arrow[d, tail] \arrow[dd, bend right=60, "\varphi_f{}^*(\sigma_{v'\cdot v})"'] \\
D_{v\cdot f} \arrow[d, tail, "\varphi_f{}^*(\sigma_v)"] \\
D_f \arrow[r, "K(u'\cdot u,v'\cdot v)"'] & D_h,
\end{tikzcd}
\tag{5.iv}
$$

where s and s' are the admissible monomorphisms that yield the 2-cells in (u,v) and (u',v') respectively and s'' is the admissible monomorphism that yields the 2-cell $h \cdot u' \cdot u \leq v' \cdot v \cdot f$. First of all we observe that

$$\sigma_h{}^*\left(\varphi_{u'\cdot u}\right) = \sigma_h{}^*\left(\varphi_{u'}\right)\cdot b,$$

where b is obtained by the following pullback

$$\begin{tikzcd}
D_{h\cdot u'\cdot u} \arrow[r, "b"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & D_{h\cdot u'} \arrow[r, "\sigma_h{}^*(\varphi_{u'})"] \arrow[d, tail, "\varphi_{u'}{}^*(\sigma_h)"] \arrow[dr, phantom, "\lrcorner", very near start] & D_h \arrow[d, tail, "\sigma_h"] \\
D_{u'\cdot u} \arrow[r, "\sigma_{u'}{}^*(\varphi_u)"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend right, "\varphi_{u'\cdot u}"'] & D_{u'} \arrow[r, "\varphi_{u'}"] \arrow[d, tail, "\sigma_{u'}"] & A'' \\
D_u \arrow[r, "\varphi_u"'] & A',
\end{tikzcd}$$

where A' is the codomain of g and A'' is the codomain of h. Furthermore, we remark that

$$\begin{tikzcd}
D_{h\cdot u'} \arrow[rr, tail, "s'"] \arrow[d, tail, "\varphi_{u'}{}^*(\sigma_h)"'] & & D_{v'\cdot g} \arrow[d, tail, "\varphi_g{}^*(\sigma_v)"] \\
D_{u'} \arrow[dr, tail, "\sigma_{u'}"'] & & D_g \arrow[dl, tail, "\sigma_g"] \\
& A' &
\end{tikzcd}$$

depict the same equivalence class of $D_{h\cdot u'}$ as subobjects of A', and therefore pullbacks of φ_u along these arrows are equal. More explicitly we have that

$$a = \left(\sigma_g\cdot\varphi_g{}^*\left(\sigma_v\right)\cdot s'\right)^*\left(\varphi_u\right) = \left(\sigma_{u'}\cdot\varphi_{u'}{}^*\left(\sigma_h\right)\right)^*\left(\varphi_u\right) = b.$$

Therefore we have $\sigma_h{}^*\left(\varphi_{u'}\right)\cdot a = \sigma_h{}^*\left(\varphi_{u'\cdot u}\right)$ in (5.iv). Then writing explicitly the domain components of (5.iv), we need to check commutativity of

$$\varphi_f{}^*\left(\sigma_v\right)\cdot\varphi_{v\cdot f}{}^*\left(\sigma_{v'}\right)\cdot s'' = \varphi_f{}^*\left(\sigma_v\right)\cdot s\cdot\varphi_{g\cdot u}{}^*\left(\sigma_{v'}\right)\cdot\left(s' * u\right). \tag{5.v}$$

We recall that s'' is defined as

$$\begin{tikzcd}
D_{h\cdot u'\cdot u} \arrow[r, tail, "s' * u"'] \arrow[rr, tail, bend left, "s''"] & D_{v'\cdot g\cdot u} \arrow[r, tail, "v' * s"'] & D_{v'\cdot v\cdot f}
\end{tikzcd}$$

following the notation of §3.8. Hence, the equality in (5.v) is represented by the following diagram

$$\begin{tikzcd}[column sep=small]
 & & D_{g\cdot u} \arrow[dr, tail, "s"] & & \\
D_{h\cdot u'\cdot u} \arrow[r, tail, "s' * u"] & D_{v'\cdot g\cdot u} \arrow[ur, tail, "\varphi_{g\cdot u}{}^*(\sigma_{v'})"] \arrow[dr, tail, "v' * s"'] & & D_{v\cdot f} \arrow[r, tail, "\varphi_f{}^*(\sigma_v)"] & D_f . \\
 & & D_{v'\cdot v\cdot f} \arrow[ur, tail, "\varphi_{v\cdot f}{}^*(\sigma_{v'})"'] & &
\end{tikzcd}$$

We remark that the central rhombus appears in the definition of $v' * s$ and corresponds to the square in (3.vii), hence it is commutative. We conclude that $K(u'\cdot u, v'\cdot v) = K(u', v')\cdot K(u, v)$, and thus L and R constitute a lax functorial factorisation system. $\square$

Next we discuss lax predistributivity of (F, L, R, K). In order to do so we recall that Tot denotes the class of total maps in $\mathcal{P}(\mathscr{C})$. Moreover, we say that a partial morphism f is a *partial left adjoint* if $\varphi_f = \sigma\cdot\widetilde{\varphi}_f$ sor some $\sigma\in\mathcal{S}$ and some morphism $\widetilde{\varphi}_f$ that is a left adjoint in $\mathscr{C}$. We denote by PLA the class of partial left adjoints of $\mathcal{P}(\mathscr{C})$.

Proposition 5.2. *The* FFS_{lax} *described in the previous proposition is predistributive and the underlying* WFS_{lax} *is* $(\mathsf{PLA}, \mathsf{Tot})$.

Proof. We consider the two classes $\mathcal{L}_F$ and $\mathcal{R}_F$ defined in (4.xviii). Let us consider $\mathcal{L}_F$. It is immediate to notice that if f is a partial left adjoint, then $Rf = (\mathrm{id}_{D_f}, \varphi_f)$ is a left adjoint in $\mathcal{P}(\mathscr{C})$, by Proposition 3.10. Hence, by Proposition 4.4 we get that Rf is laxly weakly orthogonal to any other morphism in $\mathcal{P}(\mathscr{C})$, and in particular this yields $f \pitchfork Rf$, therefore $\mathsf{PLA}\subseteq\mathcal{L}_F$.

On the other hand, if we consider $f\in\mathcal{L}_f$, we have that $f\pitchfork Rf$. In particular we have a lax diagonal morphism δ for the square η_f

$$\begin{tikzcd}
A \arrow[r, "Lf"] \arrow[d, "f"'] \arrow[dr, phantom, "\geq", near start] & D_f \arrow[d, "Rf"] \\
B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "\delta"] \arrow[ur, phantom, "\geq", near end, shift right=3] & B .
\end{tikzcd}$$

Then we look at the explicit diagrams defining the two 2-cells

$$\begin{tikzcd}[column sep=small]
D_f \arrow[dr, tail, "s"] \arrow[dddr, bend right, "\sigma_f"'] \arrow[dddrrr, bend left, "\mathrm{id}_{D_f}"] & & & \\
 & D_{\delta\cdot f} \arrow[d, tail] \arrow[dr] \arrow[r, phantom, "\cong"] & & \\
 & D_f \arrow[d, tail, "\sigma_f"'] \arrow[dr, "\varphi_f"] & D_\delta \arrow[d, tail, "\sigma_\delta"'] \arrow[dr, "\varphi_\delta"] & \\
 & A \arrow[r, "f"', "\shortmid" marking] & B \arrow[r, "\delta"', "\shortmid" marking] & D_f .
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
B \arrow[dddr, bend right, "\mathrm{id}_B"'] \arrow[dddrrr, bend left, "\mathrm{id}_B"] & & & \\
 & D_{Rf\cdot\delta} \arrow[ul, tail, "s'"'] \arrow[d, tail] \arrow[dr, "\varphi_\delta"] \arrow[r, phantom, "\leq"] & & \\
 & D_\delta \arrow[d, tail, "\sigma_\delta"'] \arrow[dr, "\varphi_\delta"] & D_f \arrow[d, tail, "\mathrm{id}_{D_f}"] \arrow[dr, "\varphi_f"] & \\
 & B \arrow[r, "\delta"', "\shortmid" marking] & D_f \arrow[r, "Rf"', "\shortmid" marking] & B .
\end{tikzcd}
\tag{5.vi}$$

Looking at the left hand diagram we notice that $\sigma_f \cdot \varphi_f{}^*(\sigma_\delta) \cdot s = \sigma_f$ yields that $\varphi_f{}^*(\sigma_\delta)$ and s are isomorphisms and therefore $\varphi_f = \sigma_\delta \cdot \sigma_\delta{}^*(\varphi_f)$. We denote by $\widetilde{\varphi}_f$ the morphism $\sigma_\delta{}^*(\varphi_f)$. In particular we have in $\mathscr{C}$ the following 2-cells

$$\begin{cases} \mathrm{id}_{D_f} \sqsubseteq \varphi_\delta \cdot \widetilde{\varphi}_f \\ \widetilde{\varphi}_f \cdot \varphi_\delta \sqsubseteq \mathrm{id}_{D_\delta}. \end{cases}$$

The first 2-cell follows from the left diagram in (5.vi) remarking that s is an isomorphism. The second 2-cell is obtained from the 2-cell $\varphi_f \cdot \varphi_\delta = \sigma_\delta \cdot \widetilde{\varphi}_f \cdot \varphi_\delta \sqsubseteq s' = \sigma_\delta$ in the right hand diagram, recalling that σ_δ belongs to $\mathcal{S}$, and is thus full. This implies that f is a partial left adjoint and thus $\mathcal{L}_F = \mathsf{PLA}$.

As for $\mathcal{R}_F$ we remark that it contains all total maps. In fact, if $f : A \longrightarrow B$ is total, then $Lf = (\mathrm{id}_A, \mathrm{id}_A)$ is an isomorphism, so $Lf \pitchfork f$. On the other hand, given $f \in \mathcal{R}_F$, we have that there exists a lax diagonal morphism δ' for ε_f

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "Lf"'] \arrow[dr, phantom, "\leq", near start] & A \arrow[d, "f"] \\ Kf \arrow[ur, "\delta'"] \arrow[r, "Rf"'] \arrow[ur, phantom, "\leq", near end, shift right=3] & B. \end{tikzcd}$$

In particular $\mathrm{id}_A \leq \delta' \cdot Lf$ yields that $\delta' \cdot Lf$ is total by Proposition 3.9 and thus Lf is total by Proposition 3.1 and $\sigma_f = \mathrm{id}_A$. Therefore $\mathcal{R}_F = \mathsf{Tot}$.

Using the new characterizations of $\mathcal{L}_F$ and $\mathcal{R}_F$ given above, to prove lax predistributivity it is enough to notice that, for every partial morphism f, $Lf = (\sigma_f, \mathrm{id}_{D_f}) \in \mathsf{PLA}$ and $Rf = (\mathrm{id}_{D_f}, \varphi_f) \in \mathsf{Tot}$. We conclude that $(\mathsf{PLA}, \mathsf{Tot})$ is the lax weak factorisation system underlying the lax functorial factorisation system above. □

Then we have that Corollary 3.11 allows us to state the following consequence of the previous proposition.

Corollary 5.3. *If the Ord-enrichment on $\mathscr{C}$ is discrete, then the* WFS$_{\mathrm{lax}}$ *underlying the* FFS$_{\mathrm{lax}}$ *in* (5.i) *is* $(\overline{\mathcal{S}}, \mathsf{Tot})$*, where* $\overline{\mathcal{S}} = \{f | \varphi_f \in \mathcal{S}\}$.

This is due to the fact that if $\mathscr{C}$ is a discrete category, then the only adjoint pairs are isomorphisms and thus PLA contains those morphisms such that $\varphi_f \in \mathcal{S}$.

5.2 Oplax WFS among partial maps and $\mathcal{S}$-stable WFS

This section is dedicated to describe the close interplay between factorisation systems on a category $\mathscr{C}$ and oplax factorisation systems on the category of partial maps $\mathcal{P}(\mathscr{C})$. In the first subsection we present a procedure that extends a stable WFS$_{\mathrm{oplax}}$ on $\mathscr{C}$ to an WFS$_{\mathrm{oplax}}$ on $\mathcal{P}(\mathscr{C})$. Then we show that, if the **Ord**-enrichment comes from a discrete partial order on $\mathscr{C}$, functoriality is transferred to the factorisation system among partial maps. Thereafter we proceed to analyse how WFSs$_{\mathrm{oplax}}$ on partial maps may be restricted to WFSs among total maps. We will conclude observing that these two processes yield a bijective correspondence between oplax weak factorisation systems on $\mathcal{P}(\mathscr{C})$ and stable WFSs on $\mathscr{C}$.

5.2.1 Oplax WFSs from total maps to partial maps

Let $\mathscr{C}$ be an **Ord**-category and $(\mathcal{E},\mathcal{M})$ an WFS$_{\text{oplax}}$ such that $\mathcal{E}$ is $\mathcal{S}$-stable, namely it is stable under pullbacks along morphisms in $\mathcal{S}$. Considering $\mathcal{P}(\mathscr{C})$, each partial map f can be factorised as

$$\begin{tikzcd}
D_f \arrow[r, "e_{\varphi_f}"] \arrow[d, tail, "\sigma_f"'] \arrow[dr, "\varphi_f"] & M_f \arrow[d, "m_{\varphi_f}"] \\
A \arrow[r, "f"', "|" marking] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
D_f \arrow[d, tail, "\sigma_f"'] \arrow[dr, "e_{\varphi_f}"] & M_f \arrow[d, tail, "\mathrm{id}_{M_f}"] \arrow[dr, "m_{\varphi_f}"] & \\
A \arrow[r, "e_f"', "|" marking] & M_f \arrow[r, "m_f"', "|" marking] & B.
\end{tikzcd} \tag{5.vii}$$

We consider the following classes of partial morphisms

$$\overline{\mathcal{E}} = \left\{ f | \varphi_f \in \mathcal{E} \right\} \qquad\qquad \overline{\mathcal{M}} = \left\{ f | \varphi_f \in \mathcal{M} \right\}. \tag{5.viii}$$

Our goal is to prove that such factorisation constitutes an WFS$_{\text{oplax}}$. We start by checking that $\overline{\mathcal{E}} \pitchfork \overline{\mathcal{M}}$. We consider $f \in \overline{\mathcal{E}}$, $g \in \overline{\mathcal{M}}$ and the oplax square $(u,v) : f \longrightarrow g$. Writing explicitly the oplax square, one remarks that, since $\mathcal{E}$ is stable under pullbacks along morphisms in $\mathcal{S}$, then $\sigma_v{}^*\left(\varphi_f\right)$ belongs to $\mathcal{E}$. Since $(\mathcal{E},\mathcal{M})$ is an WFS$_{\text{oplax}}$, then $\mathcal{E} \pitchfork \mathcal{M}$. Thus there exists an oplax diagonal filler d for the following oplax square in $\mathscr{C}^2_{\text{oplax}}$

$$\begin{tikzcd}
D_{v\cdot f} \arrow[r, tail, "s"] \arrow[d, "\mathcal{E}\ni\sigma_v{}^*(\varphi_f)"'] \arrow[dr, phantom, "\sqsubseteq"] & D_{g\cdot u} \arrow[r, "\sigma_g{}^*(\varphi_u)"] \arrow[dr, phantom, "\sqsubseteq"] & D_g \arrow[d, tail, "\varphi_g\in\mathcal{M}"] \\
D_v \arrow[urr, dashed, "\exists d"] \arrow[rr, "\varphi_v"'] & & D,
\end{tikzcd} \tag{5.ix}$$

where s is the morphism in $\mathcal{S}$ that yields the 2-cell $v \cdot f \leq g \cdot u$. Then it is easy to check that

$$\begin{tikzcd}
D_v \arrow[d, tail, "\sigma_v"'] \arrow[dr, "\sigma_g\cdot d"] & \\
B \arrow[r, "\delta"', "|" marking] & C
\end{tikzcd} \tag{5.x}$$

is an oplax diagonal morphism for the oplax square

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\sqsubseteq"] & C \arrow[d, "g"] \\
B \arrow[ur, dotted, "\delta"] \arrow[r, "v"'] & D
\end{tikzcd}$$

In fact, the existence of the 2-cells in the upper diagram is deduced by

$$\begin{cases} \sigma_f \cdot \varphi_f{}^*(\sigma_v) = \sigma_u \cdot \varphi_u{}^*(\sigma_g) \cdot s; \\ \sigma_g \cdot d \cdot \sigma_v{}^*(\varphi_f) \sqsubseteq \sigma_g \cdot \sigma_g{}^*(\varphi_u) \cdot s = \varphi_u \cdot \varphi_u{}^*(\sigma_g) \cdot s \end{cases} \quad \Rightarrow \quad \begin{tikzcd} D_u \arrow[dddr, bend right, "\sigma_u"'] \arrow[dddrrr, bend left, "\varphi_u"] & & & \\ & D_{v\cdot f} \arrow[ul, "\varphi_u{}^*(\sigma_g)\cdot s"'] \arrow[d, "\varphi_f{}^*(\sigma_v)"'] \arrow[dr, "\sigma_v{}^*(\varphi_f)"] \arrow[drr, phantom, "\sqsubseteq"] & & \\ & D_f \arrow[d, "\sigma_f"'] \arrow[dr, "\varphi_f"] & D_v \arrow[d, "\sigma_v"'] \arrow[dr, "\sigma_g\cdot d"] & \\ & A \arrow[r, "f"'] & B \arrow[r, "\delta"'] & C. \end{tikzcd}$$

On the other hand, the existence of the 2-cell in the lower triangle is clearly shown in the following diagram

$$\begin{tikzcd} D_v \arrow[dr, "d"] \arrow[dd, "\mathrm{id}_{D_v}"'] \arrow[dddd, bend right, "\sigma_v"'] \arrow[ddddrrr, bend left, "\varphi_v"] & & & \\ & D_g \arrow[dr, "\mathrm{id}_{D_g}"] \arrow[dd, "\mathrm{id}_{D_g}"] \arrow[drr, phantom, "\cong"] & & \\ D_v \arrow[dr, "d"] \arrow[dd, "\sigma_v"'] & & D_g \arrow[dd, "\sigma_g"] \arrow[ddr, "\varphi_g"] & \\ & D_g \arrow[dr, "\sigma_g"] & & \\ B \arrow[rr, "\delta"'] & & C \arrow[r, "g"'] & D. \end{tikzcd}$$

Since the morphism that yields the 2-cell $v \le g \cdot \delta$ is an isomorphism, we remark that if the **Ord**-enrichment $\sqsubseteq$ is discrete, then $v = g \cdot \delta$. This allows us to conclude that $\overline{\mathcal{E}} \pitchfork \overline{\mathcal{M}}$. Furthermore, let us consider a morphism $f : A \longrightarrow B$ belonging to ${}^{\pitchfork}\overline{\mathcal{M}}$. In particular this implies that $f \pitchfork m_f$, and thus the (commutative) square $(e_f, \mathrm{id}_B) : f \longrightarrow m_f$ admits an oplax diagonal morphism δ. We observe that, by Proposition 3.9 and Proposition 3.1, $\mathrm{id}_B \le m_f \cdot \delta$ yields that δ is a total morphism. We consider the oplax square in $\mathscr{C}$ $(e_{\varphi_f}, \mathrm{id}_B) : \varphi_f \longrightarrow m_{\varphi_f}$ and we have the following 2-cells

$$\begin{cases} \delta \cdot f \le e_f \\ \mathrm{id}_B \le m_f \cdot \delta, \end{cases} \qquad \Rightarrow \qquad \begin{cases} \varphi_\delta \cdot \varphi_f \sqsubseteq e_{\varphi_f} \\ \mathrm{id}_B \sqsubseteq m_{\varphi_f} \cdot \varphi_\delta. \end{cases}$$

Hence φ_δ is an oplax diagonal lifting for the oplax square

$$\begin{tikzcd} D_f \arrow[r, "e_{\varphi_f}"] \arrow[d, "\varphi_f"'] \arrow[dr, phantom, "\sqsupseteq", near start] & M_f \arrow[d, "m_{\varphi_f}"] \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, dotted, "\varphi_\delta"] \arrow[ur, phantom, "\sqsupseteq", near end, shift right=3] & B. \end{tikzcd}$$

Then we consider an oplax square $(u,v) : \varphi_f \longrightarrow g$ in $\mathscr{C}^2_{\text{oplax}}$ with $g \in \mathcal{M}$. Hence, by dual arguments to those already discussed in the proof of Theorem 4.26, we adapt diagram (4.xxvii) and we obtain

$$\begin{tikzcd}
D_f \arrow[r, "u"] \arrow[d, "e_{\varphi_f}"'] \arrow[dr, phantom, "\Leftarrow"] & C \arrow[d, "e_g"] \\
M_f \arrow[r, "K(u,v)"] \arrow[d, "m_{\varphi_f}"'] \arrow[dr, phantom, "\Leftarrow"] & M_g \arrow[d, "m_g"] \arrow[u, bend right=60, "\lambda_g"'] \\
B \arrow[r, "v"'] \arrow[u, bend left=60, "\varphi_\delta"] & D.
\end{tikzcd}$$

and, always by arguments analogous to those in such proof, we get that $\lambda_g \cdot K(u,v) \cdot \varphi_\delta$ is an oplax diagonal morphism for the oplax square (u,v). This allows us to conclude that $\varphi_f \pitchfork \mathcal{M}$, and therefore $f \in \overline{\mathcal{E}}$.

In a similar fashion one can prove that if $g \in {}^{\pitchfork}\overline{\mathcal{E}}$, then the square $(\mathrm{id}_A, m_g) : e_g \longrightarrow g$ admits an oplax diagonal lifting δ'. Again, since m_g is total, the 2-cell $m_g \leq g \cdot \delta'$ yields that δ' is total. Writing explicitly such 2-cell we obtain the following diagram

$$\begin{tikzcd}
M_g \arrow[d, tail, "\mathrm{id}_{M_g}"] \arrow[dr, "\widetilde{\varphi}_{\delta'}"] \arrow[ddrr, bend left=40, "m_{\varphi_g}"] \arrow[dd, bend right=70, "\mathrm{id}_{M_g}"'] & & \\
M_g \arrow[d, tail, "\mathrm{id}_{M_g}"] \arrow[dr, "\varphi_{\delta'}"] & D_g \arrow[d, tail, "\sigma_g"] \arrow[dr, "\varphi_g"] \arrow[ur, phantom, "\geq"] & \\
M_g \arrow[r, "\delta'"', "|" marking] & A \arrow[r, "g"', "|" marking] & B.
\end{tikzcd}$$

In particular, we observe that $\varphi_{\delta'} = \sigma_g \cdot \widetilde{\varphi}_{\delta'}$ and $m_{\varphi_g} \sqsubseteq \varphi_g \cdot \widetilde{\varphi}_{\delta'}$. Then $\delta' \cdot e_g \leq \mathrm{id}_A$ yields $\sigma_g \cdot \widetilde{\varphi}_{\delta'} \cdot e_{\varphi_{\delta'}} = \varphi_{\delta'} \cdot e_{\varphi_{\delta'}} \sqsubseteq \sigma_g$, and thus $\widetilde{\varphi}_{\delta'} \cdot e_{\varphi_{\delta'}} \sqsubseteq \mathrm{id}_{D_g}$, since morphisms in $\mathcal{S}$ are full. This yields that $\widetilde{\varphi}_{\delta'}$ is an oplax diagonal lifting for the square $\left(\mathrm{id}_A, m_{\varphi_g}\right) : e_{\varphi_g} \longrightarrow g$. Hence, we conclude by similar arguments as before that $\varphi_g \in \mathcal{E}^{\pitchfork} = \mathcal{M}$, thus $g \in \overline{\mathcal{M}}$.

The construction that we have presented in this section can be resumed in the following result.

Proposition 5.4. *Let $\mathscr{C}$ be an* **Ord***-category, $\mathcal{S}$ a class of admissible morphisms full and upper-closed and $(\mathcal{E}, \mathcal{M})$ an $\mathcal{S}$-stable* WFS$_{\text{oplax}}$*. Then $\left(\overline{\mathcal{E}}, \overline{\mathcal{M}}\right)$ is an* WFS$_{\text{oplax}}$ *on $\mathcal{P}(\mathscr{C})$.*

Example 5.5. *Let us consider $\mathcal{P}$(***Set***). We know that in* **Set** *the two classes* Epi *and* Mono *are stable under pullbacks. Moreover,* (Mono, Epi) *is a stable* WFS*, as presented in Example 2.11. Then by the arguments discussed above, we deduce that* $\left(\overline{\mathsf{Mono}}, \overline{\mathsf{Epi}}\right)$ *is an* WFS$_{\text{oplax}}$ *for $\mathcal{P}$(***Set***). Furthermore, we have that* (Epi, Mono) *is a stable* OFS*. This yields that* $\left(\overline{\mathsf{Epi}}, \overline{\mathsf{Mono}}\right)$ *is an* WFS$_{\text{oplax}}$.

Remark 5.6. *As for* **Set***, we pointing out that* $\left(\overline{\mathsf{Mono}}, \overline{\mathsf{Epi}}\right)$*-factorisations are not unique. In fact, we know that every non-empty morphism in* **Set** *has multiple* (Mono, Epi)*-factorisations (actually infinite), as already remarked in Example 2.11. Thus, we observe that from the construction* (5.vii) *each of these factorisations gives rise to a different factorisation on a partial map.*

5.2.2 Functoriality from total factorisation systems to oplax WFS

In this section we discuss how the structure on a WFS on $\mathscr{C}$ is transferred to the $\mathrm{WFS}_{\mathrm{oplax}}$ on $\mathcal{P}(\mathscr{C})$ yielded by the construction in the previous section.

We assume along this section that the **Ord**-enrichment $\sqsubseteq$ on $\mathscr{C}$ is discrete, which implies that being an $\mathrm{OFS}_{\mathrm{oplax}}$ is equivalent to being a WFS. Under this assumption we will show that, if the WFS is orthogonal, then the induced $\mathrm{WFS}_{\mathrm{oplax}}$ on $\mathcal{P}(\mathscr{C})$ admits an oplax functorial realisation.

Proposition 5.7. *Let $\mathscr{C}$ be an ordinary category and $(\mathcal{E},\mathcal{M})$ a stable* WFS *on $\mathscr{C}$ that admits the functorial realisation (F,L,R,K). Then the* $\mathrm{WFS}_{\mathrm{oplax}}$ *admits an oplax functorial realisation $\left(\overline{F},\overline{L},\overline{R},\overline{K}\right)$.*

Proof. We observe that any morphism $f:A\longrightarrow B$ in $\mathcal{P}(\mathscr{C})$ admits the $\left(\overline{\mathcal{E}},\overline{\mathcal{M}}\right)$-factorisation described in diagram (5.vii). Our goal is to prove that such factorisation is functorial. We adopt the following notation

$$\overline{L}f=\left(\sigma_f,L\varphi_f\right) \qquad \overline{R}f=\left(\mathrm{id}_{K\varphi_f},R\varphi_f\right) \qquad \overline{K}f=K\varphi_f$$

We consider two composable oplax squares

$$\begin{tikzcd}
A \arrow[r, "u", "\shortmid"{marking}] \arrow[d, "f"', "\shortmid"{marking}] \arrow[dr, phantom, "\swarrow"] & C \arrow[r, "u'", "\shortmid"{marking}] \arrow[d, "g", "\shortmid"{marking}] \arrow[dr, phantom, "\swarrow"] & E \arrow[d, "h", "\shortmid"{marking}] \\
B \arrow[r, "v"', "\shortmid"{marking}] & D \arrow[r, "v'"', "\shortmid"{marking}] & G.
\end{tikzcd}$$

We aim to define $\overline{K}$ and to prove that $\overline{K}(u'\cdot u,v'\cdot v)=\overline{K}(u',v')\cdot\overline{K}(u,v)$.

We consider the following oplax square

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "\overline{L}f"'] & C \arrow[r, "\overline{L}g"] \arrow[d, phantom, "\swarrow"] & \overline{K}g \arrow[d, "\overline{R}g"] \\
\overline{K}f \arrow[r, "\overline{R}f"'] & B \arrow[r, "v"'] & D.
\end{tikzcd}$$

We apply to this diagram the construction in diagram (5.ix), which, in this case, is commutative, since the **Ord**-enrichment on $\mathscr{C}$ is discrete. This yields the following diagram

$$\begin{tikzcd}
D_{v\cdot f} \arrow[r, tail, "s"] \arrow[d, "\sigma_{v\cdot\overline{R}f}{}^{*}\left(\varphi_{\overline{L}f}\right)"'] & D_{g\cdot u} \arrow[r, "\varphi_{\overline{L}g\cdot u}"] & \overline{K}g \arrow[d, "R\varphi_g"] \\
D_{v\cdot\overline{R}f} \arrow[rr, "\varphi_{v\cdot\overline{R}f}"'] \arrow[urr, "k"] & & D,
\end{tikzcd} \tag{5.xi}$$

where we choose k, among the possible diagonal lifting of such square, as

$$k=K\left(\varphi_{\overline{L}g\cdot u}\cdot s,\varphi_{v\cdot\overline{R}f}\right)=K\left(L\varphi_g\cdot\sigma_g{}^{*}\left(\varphi_u\right)\cdot s,\varphi_v\cdot\sigma_v{}^{*}\left(R\varphi_f\right)\right). \tag{5.xii}$$

The morphism k constitute a diagonal lifting for (5.xi) since $(\mathcal{E},\mathcal{M})$ is an OFS. In fact this yields that for any $f \in \mathcal{E}$ and $g \in \mathcal{M}$, then Rf and Lg are isomorphism. Then $\overline{K}(u,v) = \left(\left(R\varphi_f\right)^*(\sigma_v),k\right)$, as defined in (5.x). Similarly we have $\overline{K}(u',v') = \left((R\varphi_g)^*(\sigma_{v'}),k'\right)$, where

$$k' = K\left(\varphi_{\overline{L}h\cdot u'}\cdot s', \varphi_{v'\cdot\overline{R}g}\right) = K\left(L\varphi_h\cdot{\sigma_h}^*(\varphi_{u'})\cdot s', \varphi_{v'}\cdot{\sigma_{v'}}^*(R\varphi_g)\right). \tag{5.xiii}$$

Finally we define analogously $\overline{K}(u'\cdot u, v'\cdot v) = \left(\sigma_{v'\cdot v\cdot\overline{R}f}, k''\right)$. We remark that

$$\sigma_{v'\cdot v\cdot\overline{R}f} = \left(R\varphi_f\right)^*(\sigma_v)\cdot\left[\left(\sigma_v^*\left(R\varphi_f\right)\right)^*\left({\varphi_v}^*(\sigma_{v'})\right)\right] = \left(R\varphi_f\right)^*(\sigma_v)\cdot\left[\left(\varphi_v\cdot{\sigma_v}^*\left(R\varphi_f\right)\right)^*(\sigma_{v'})\right],$$

which we deduce from the following diagram, recalling that $\overline{R}f$ is total by definition,

$$\begin{tikzcd}
D_{v'\cdot v\cdot f} \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & D_{v'\cdot v} \arrow[r, "{\sigma_{v'}}^*(\varphi_v)"] \arrow[d, tail, "{\varphi_v}^*(\sigma_{v'})"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_{v'} \arrow[d, tail, "\sigma_{v'}"] \\
D_{v\cdot f} \arrow[r, "{\sigma_v}^*(R\varphi_f)"'] \arrow[d, tail, "R{\varphi_f}^*(\sigma_v)"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_v \arrow[r, "\varphi_v"'] \arrow[d, tail, "\sigma_v"'] & D \\
K\varphi_f \arrow[r, "R\varphi_f"'] & B. &
\end{tikzcd}$$

Again we point out that k'' is the diagonal morphism chosen through K for the commutative square

$$\begin{tikzcd}
D_{v'\cdot v\cdot f} \arrow[r, tail, "s*s'"] \arrow[d, "\left(\sigma_{v'\cdot v\cdot\overline{R}f}\right)^*\left(\varphi_{\overline{L}f}\right)"'] & D_{h\cdot u'\cdot u} \arrow[r, "\varphi_{\overline{L}h\cdot u'\cdot u}"] & \overline{K}h \arrow[d, "R\varphi_h"] \\
D_{v'\cdot v\cdot\overline{R}f} \arrow[urr, "k''"] \arrow[rr, "\varphi_{v'\cdot v\cdot\overline{R}f}"'] & & F.
\end{tikzcd} \tag{5.xiv}$$

We write explicitly the composition $\overline{K}(u',v')\cdot\overline{K}(u,v)$

$$\begin{tikzcd}
P \arrow[d, tail, "k^*(R{\varphi_g}^*(\sigma_{v'}))"'] \arrow[dr, "(R{\varphi_g}^*(\sigma_{v'}))^*(k)"] & & \\
D_{v\cdot\overline{R}f} \arrow[d, tail, "R{\varphi_f}^*(\sigma_v)"'] \arrow[dr, "k"] & D_{v'\cdot\overline{R}g} \arrow[d, tail, "R{\varphi_g}^*(\sigma_{v'})"'] \arrow[dr, "k'"] & \\
\overline{K}f \arrow[r, "\overline{K}(u,v)"', "\shortmid" marking] & \overline{K}g \arrow[r, "\overline{K}(u',v')"', "\shortmid" marking] & \overline{K}h.
\end{tikzcd} \tag{5.xv}$$

From (5.xi) we know that $\varphi_{v \cdot \overline{R}f} = \varphi_v \cdot \sigma_v{}^* \left(R\varphi_f \right) = R\varphi_g \cdot k$. Hence we have the equality of the two domains

$$
\begin{aligned}
R\varphi_f{}^* \left(\sigma_v \right) \cdot k^* \left(R\varphi_g{}^* \left(\sigma_{v'} \right) \right) &= R\varphi_f{}^* \left(\sigma_v \right) \cdot \left[\left(R\varphi_g \cdot k \right)^* \left(\sigma_{v'} \right) \right] \\
&= R\varphi_f{}^* \left(\sigma_v \right) \cdot \left[\left(\varphi_v \cdot \sigma_v{}^* \left(R\varphi_f \right) \right)^* \left(\sigma_{v'} \right) \right] \\
&= R\varphi_f{}^* \left(\sigma_v \right) \cdot \varphi^*_{v \cdot \overline{R}f} \left(\sigma_{v'} \right).
\end{aligned}
$$

Thus we can write $P = D_{v' \cdot v \cdot \overline{R}f}$, since it is the domain of $\varphi^*_{v \cdot \overline{R}f} \left(\sigma_{v'} \right)$.

Now we turn our attention to the function component of the partial maps. We consider the following diagram

$$
\begin{tikzcd}
D_{v' \cdot v \cdot f} \arrow[r, tail, "v' * s"] \arrow[d, "\left(\sigma_{v' \cdot v \cdot \overline{R}f} \right)^* \left(L\varphi_f \right)"'] & D_{v' \cdot g \cdot u} \arrow[r, "\sigma_{v' \cdot g}{}^* \left(\varphi_u \right)"] \arrow[d, phantom, "\boxed{1}"] & D_{v' \cdot g} \arrow[r, tail, "s'"] \arrow[d, "\left(R\varphi_g{}^* \left(\sigma_{v'} \right) \right)^* \left(L\varphi_g \right)"] & D_{h \cdot u'} \arrow[r, "\varphi_{\overline{L}h \cdot u'}"] \arrow[d, phantom, "\boxed{2}"] & \overline{K}h \arrow[d, "R\varphi_h"] \\
D_{v' \cdot v \cdot \overline{R}f} \arrow[rr, "\left(R\varphi_g{}^* \left(\sigma_{v'} \right) \right)^* (k)"'] & & D_{v' \cdot \overline{R}g} \arrow[r, "\sigma_{v'}{}^* \left(R\varphi_g \right)"'] & D_{v'} \arrow[r, "\varphi_{v'}"'] & F.
\end{tikzcd}
$$

We highlight that the diagram and its subsquares are commutative due to the properties of pullbacks and the definition of each morphism. In particular we notice that $\boxed{2}$ is the square that yields the definition of k' in (5.xiii). We also remark that $\left(\sigma_{v' \cdot v \cdot \overline{R}f} \right)^* \left(L\varphi_f \right)$ and $\left(R\varphi_g{}^* \left(\sigma_{v'} \right) \right)^* \left(L\varphi_g \right)$ are morphisms in $\mathcal{E}$, hence the factorisation of the square $\boxed{1}$ yields that

$$
K \left(\sigma_{v' \cdot g}{}^* \left(\varphi_u \right) \cdot \left(v' * s \right), \left(R\varphi_g{}^* \left(\sigma_{v'} \right) \right)^* (k) \right) = \left(R\varphi_g{}^* \left(\sigma_{v'} \right) \right)^* (k).
$$

We focus on the outer square and we claim that it is exactly the square in (5.xiv), which is used to define k'' as a diagonal filler obtained through the functor K. This claim is trivially satisfied by the vertical edges. Recalling that $\varphi_{\overline{L}h \cdot u' \cdot u} = \varphi_{\overline{L}h \cdot u'} \cdot \sigma_{u'}{}^* \left(\varphi_{h \cdot u} \right)$, we point out that the equality of the top edges is yielded by the composition of 2-cells in $\mathcal{P}(\mathscr{C})$. Furthermore, looking at the bottom edges, we point out that

$$
\varphi_{v'} \cdot \sigma_{v'}{}^* \left(R\varphi_g \right) \cdot \left(R\varphi_g{}^* \left(\sigma_{v'} \right) \right)^* (k) = \varphi_{v'} \cdot \sigma_{v'}{}^* \left(R\varphi_g \cdot k \right) = \varphi_{v'} \cdot \sigma_{v'}{}^* \left(\varphi_{v \cdot \overline{R}f} \right) = \varphi_{v' \cdot v \cdot \overline{R}f},
$$

where the first equality follows from the properties of pullbacks, the second by the definition of k given in (5.xii) and the last follows from the definition of composition of partial maps.

Thus, since k'' is chosen through K, by functoriality we can conclude that

$$
k'' = k' \cdot \left(R\varphi_g{}^* \left(\sigma_{v'} \right) \right)^* (k),
$$

which is equal to the function component of $\overline{K} \left(u', v' \right) \cdot \overline{K} \left(u, v \right)$, as shown in (5.xv). This was the last piece needed to conclude that $\left(\overline{F}, \overline{L}, \overline{R}, \overline{K} \right)$ is an oplax functorial factorisation system. $\square$

We remark that the properties that are transferred from $\left(\mathcal{E}, \mathcal{M} \right)$ to $\left(\overline{\mathcal{E}}, \overline{\mathcal{M}} \right)$ go beyond extending its functorial realisation.

Theorem 5.8. *If $(\mathcal{E},\mathcal{M})$ is an $\mathcal{S}$-stable* OFS*, then $\left(\overline{L},\overline{R},\overline{K}\right)$ extends to an* $\text{AWFS}_{\text{oplax}}$ *on $\mathcal{P}(\mathscr{C})$.*

Proof. By the previous proposition, we have that $\left(\overline{L},\overline{R},\overline{K}\right)$ gives rise to the oplax copointed endofunctor $\left(\overline{L},\overline{\varepsilon}\right)$ and the oplax pointed endofunctor $\left(\overline{R},\overline{\eta}\right)$ on $\mathcal{P}(\mathscr{C})$. Considering the comonad $\mathbb{L} = (L,\varepsilon,\Omega)$ and the monad $\mathbb{R} = (R,\eta,\Theta)$, our goal is to extend the two natural transformations Θ and Ω.

We focus on $\mathbb{L}$. We notice that (L,ε) already has the structure of a lax copointed endofunctor, so we aim to extend Ω. Given a partial morphism $f : A \longrightarrow B$, we define $\overline{\Omega}_f$ as

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\overline{L}f"'] \arrow[dr, phantom, "\overline{\Omega}_f"] & A \arrow[d, "\overline{LL}f"] \\
\overline{K}f \arrow[r, "\overline{\omega}_f"'] \arrow[d, equal] & \overline{KL}f \arrow[d, equal] \\
K\varphi_f \arrow[r, "\omega_{\varphi_f}"'] & KL\varphi_f,
\end{tikzcd}$$

where $\overline{\omega}_f$ is a partial morphism defined by the span $\left(\mathrm{id}_{K\varphi_f},\omega_{\varphi_f}\right)$ and ω_{φ_f} is the codomain component of Ω_{φ_f}. We remark that $\overline{\Omega}_f$ is trivially a commutative square. We aim to prove that, for any oplax square $(u,v) : f \longrightarrow g$

$$\begin{tikzcd}
\overline{L}f \arrow[r, "\overline{L}(u{,}v)"] \arrow[d, "\overline{\Omega}_f"'] \arrow[dr, phantom, "\swarrow"] & \overline{L}g \arrow[d, "\overline{\Omega}_g"] \\
\overline{LL}f \arrow[r, "\overline{LL}(u{,}v)"'] & \overline{LL}g,
\end{tikzcd}$$

in order to prove that $\overline{\Omega}$ is a lax natural transformation. We consider the following diagram

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dr, "\mathrm{id}_A"] \arrow[dd, "\overline{L}f"'] & & C \arrow[dr, "\mathrm{id}_C"] \arrow[dd, "\overline{L}g" near end] & \\
& A \arrow[rr, crossing over, "u" near start] & & C \arrow[dd, "\overline{LL}g"] \\
K\varphi_f \arrow[rr, "K(u{,}v)" near end] \arrow[dr, "\overline{\omega}_f"'] & & K\varphi_g \arrow[dr, "\overline{\omega}_g"] & \\
& KL\varphi_f \arrow[rr, "K(u{,}K(u{,}v))"'] \arrow[from=uu, crossing over, "\overline{LL}f"' near start] & & KL\varphi_g.
\end{tikzcd} \tag{5.xvi}$$

Trivially the top square is commutative. We focus on

$$\begin{tikzcd}
K\varphi_f \arrow[r, "K(u{,}v)"] \arrow[d, "\overline{\omega}_f"'] & K\varphi_g \arrow[d, "\overline{\omega}_g"] \\
KL\varphi_f \arrow[r, "K(u{,}K(u{,}v))"'] & KL\varphi_g.
\end{tikzcd}$$

Since $(\mathcal{E}, \mathcal{M})$ is orthogonal, then ω_{φ_f} and ω_{φ_g} are isomorphisms. Hence $\overline{\omega_{\varphi_f}}$ and $\overline{\omega_{\varphi_g}}$ are isomorphism as well. Thus the square is trivially commutative.

Moreover, we have the following diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "Lf"'] \arrow[dr, phantom, "\overline{\Omega}_f"] & A \arrow[r, "\mathrm{id}_A"] \arrow[d, "LLf"] \arrow[dr, phantom, "\overline{\varepsilon}_{\overline{L}f}"] & A \arrow[d, "Lf"] \\
K\varphi_f \arrow[r, "\omega_{\varphi_f}"'] & KL\varphi_f \arrow[r, "RL\varphi_f"'] & K\varphi_f
\end{tikzcd}$$

and again, by the comonad axioms, $RL\varphi_f \cdot \omega_{\varphi_f} = \mathrm{id}_{K\varphi_f}$, thus $\overline{\varepsilon}_{\overline{L}} \cdot \overline{\Omega} = \mathrm{id}_{\overline{L}}$. On the other hand we observe the following diagram

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dd, "Lf"'] & & A \arrow[rr, "\mathrm{id}_A"] \arrow[dd, "LLf"] & & A \arrow[dd, "Lf"] & \\
& D_f \arrow[ul, tail] \arrow[rr, tail] \arrow[dl] \arrow[r, phantom, "\overline{\Omega}_f"] & & D_f \arrow[ul, tail] \arrow[rr, tail] \arrow[dl] \arrow[r, phantom, "\overline{L}\overline{\varepsilon}_f"] & & D_f \arrow[ul, tail] \arrow[dl] \\
K\varphi_f \arrow[rr, "\omega_{\varphi_f}"'] & & KL\varphi_f \arrow[rr, "K\varepsilon_{\varphi_f}"'] & & K\varphi_f, &
\end{tikzcd}$$

and we remark that $K\varepsilon_{\varphi_f} \cdot \omega_{\varphi_f} = \mathrm{id}_{K\varphi_f}$, which yields $\overline{L}\overline{\varepsilon} \cdot \overline{\Omega} = \mathrm{id}_{\overline{L}}$.

Then we point out that $\overline{\Omega} \cdot \overline{L}\overline{\Omega} = \overline{\Omega} \cdot \overline{\Omega}_{\overline{L}}$ follows easily since $\overline{\Omega}_f$ is always a commutative square and its components are total.

The same arguments may be carried out for $\mathbb{R} = (R, \eta, \Theta)$ and a similar proof that $\overline{\mathbb{R}}$ is an oplax monad follows in a simpler fashion, since Rf is a total map by construction for every f. We just show the definition of $\overline{\Theta}$, which is

$$\begin{tikzcd}
KR\varphi_f \arrow[r, "\theta_{\varphi_f}"] \arrow[d, equal] & K\varphi_f \arrow[d, equal] \\
\overline{K}f\overline{R} \arrow[r, "\overline{\theta}_f"] \arrow[d, "\overline{RR}f"'] \arrow[dr, phantom, "\overline{\Theta}_f"] & \overline{K}f \arrow[d, "\overline{R}f"] \\
B \arrow[r, "\mathrm{id}_B"'] & B.
\end{tikzcd}$$

Finally, since the components of $\overline{\Omega}$ and $\overline{\Theta}$ are the same as the transformations for $\mathscr{C}$, then it is easy to check that Δ extends to a distributivity law on $\mathcal{P}(\mathscr{C})$ of $\overline{\mathbb{L}}$ over $\overline{\mathbb{R}}$ and all of the morphisms of the distributivity axioms in (4.xxxvii) are total, then its commutativity follows from the commutativity of the distributive law in $\mathscr{C}$. □

Remark 5.9. *We remark that the previous proof actually uses the fact the the functor K is a functorial lifting operator for an* OFS. *Therefore this proof can actually be extended to any functorial factorisation that admits a lifting operator, as for instance* LOFSs.

In conclusion, under the hypothesis that $\mathcal{E}$ is stable under pullbacks along morphisms in $\mathcal{S}$, an OFS $(\mathcal{E},\mathcal{M})$ on $\mathscr{C}$ induces an AWFS$_{\text{oplax}}$ $\left(\overline{\mathbb{L}},\overline{\mathbb{R}}\right)$ on $\mathcal{P}(\mathscr{C})$. Furthermore, we remark from the proof that the transformations $\overline{\Theta}$ and $\overline{\Omega}$ are indeed strict natural transformations.

Example 5.10. *We consider again the stable* OFS $(\mathsf{Epi},\mathsf{Mono})$ *for* **Set**. *In Example 5.5, we have already remarked that it gives rise to the* WFS$_{\text{oplax}}$ $\left(\overline{\mathsf{Epi}},\overline{\mathsf{Mono}}\right)$ *on* $\mathcal{P}(\mathbf{Set})$. *Hence, the arguments presented in this section allow us to conclude that* $\left(\overline{\mathsf{Epi}},\overline{\mathsf{Mono}}\right)$ *admits a functorial realisation that constitutes an* AWFS$_{\text{oplax}}$.

5.2.3 From oplax WFS on partial maps to WFS on total maps

In this section we go back to consider an **Ord**-enriched category $\mathscr{C}$ and a category of partial maps $\mathcal{P}(\mathscr{C})$. Let $(\mathcal{L},\mathcal{R})$ be an WFS$_{\text{oplax}}$. We intend to analyse what type of structure it generates on the category of total maps $\mathscr{C}$. We start this section by proving a useful result that will help us to describe WFSS$_{\text{oplax}}$ for categories of partial maps. Then we move on to study whether the orthogonality relations are preserved when restricted to total maps. Namely, our goal is to prove that any two total maps that are oplax weakly orthogonal ($f \pitchfork g$) in $\mathcal{P}(\mathscr{C})$, are oplax weakly orthogonal in $\mathscr{C}$. In particular, if the **Ord**-enrichment on $\mathscr{C}$ is discrete, this means that they are weakly orthogonal ($f \boxslash g$) in the ordinary sense.

Proposition 5.11. *Let* $(\mathcal{L},\mathcal{R})$ *be an* WFS$_{\text{oplax}}$ *on* $\mathcal{P}(\mathscr{C})$. *If* $l \in \mathcal{L}$ *and* $r \in \mathcal{R}$, *then* $(\mathrm{id},\varphi_l) \in \mathcal{L}$ *and* $(\mathrm{id},\varphi_r) \in \mathcal{R}$.

Proof. Let f be any morphism in $\mathcal{R}$. We consider an oplax square $(u,v):(\mathrm{id},\varphi_l) \longrightarrow f$. Then the factorisation in (5.i) yields that $l = (\mathrm{id}_{D_l},\varphi_l)\cdot(\sigma_l,\mathrm{id}_{D_l})$. Therefore we have that the outer rectangle in the following diagram constitute an oplax square

$$
\begin{tikzcd}
A \arrow[r, "(\sigma_l{,}\mathrm{id}_{D_l})"] \arrow[d, "l"'] & D_l \arrow[r, "u"] \arrow[d, "(\mathrm{id}_{D_l}{,}\varphi_l)"] & X \arrow[d, "f"] \\
B \arrow[r, "\mathrm{id}_B"'] \arrow[urr, dotted, "\delta"] & B \arrow[r, "v"'] & Y.
\end{tikzcd}
$$

The morphism δ is the oplax diagonal lifting of l against f. This yields the following 2-cells

$$
\begin{cases}
v \le f\cdot\delta \\
\delta\cdot(\mathrm{id}_{D_l},\varphi_l)\cdot(\sigma_l,\mathrm{id}_{D_l}) \le u\cdot(\sigma_l,\mathrm{id}_{D_l})\,.
\end{cases}
\tag{5.xvii}
$$

We point out that $(\sigma_l,\mathrm{id}_{D_l})$ is trivially a faithful morphism in $\mathcal{P}(\mathscr{C})$, hence (5.xvii) yields that δ is an oplax diagonal morphism lifting of $\left(\mathrm{id}_{D_l,\varphi_l}\right)$ against f. Thus we have that $\left(\mathrm{id}_{D_l,\varphi_l}\right) \in \mathcal{L}$.

Let g be any morphism in $\mathcal{L}$ and $(u,v) : g \longrightarrow (\mathrm{id}_{D_r},\varphi_r)$ an oplax square. We consider the following diagram

$$\begin{tikzcd}
X \arrow[rr, "u"] \arrow[dr, "u'"'] \arrow[dd, "g"'] \arrow[ddr, phantom, "\leq"] & & D_r \arrow[dd, "(\mathrm{id}_{D_r},\varphi_r)"] \\
& A \arrow[ur, "(\sigma_r,\mathrm{id}_{D_r})"'] \arrow[d, "r"] & \\
Y \arrow[ur, dotted, "\delta"] \arrow[ur, phantom, "\leq"', near end] \arrow[r, "v"'] & B \arrow[r, "\mathrm{id}_B"'] & B
\end{tikzcd}$$

where $u' = (\mathrm{id}_{D_r},\sigma_r)\cdot u$. We recall that $(\sigma_r,\mathrm{id}_{D_r})\cdot(\mathrm{id}_{D_r},\sigma_r) = \mathrm{id}_{D_r}$ trivially. Therefore it is straightforward that the upper triangle and the right hand square are commutative. Then it is easy to conclude from the diagram that $d = (\sigma_r,\mathrm{id}_{D_r})\cdot\delta$ is an oplax diagonal morphism which lifts g against $(\mathrm{id}_{D_r},\varphi_r)$. Thus $(\mathrm{id}_{D_r},\varphi_r) \in \mathcal{R}$. □

Lemma 5.12. *Let $(\mathcal{L},\mathcal{R})$ be an oplax weak factorisation system for a category of partial maps $\mathcal{P}(\mathscr{C})$. Then $(\mathcal{L}\cap\mathsf{Tot}) \pitchfork (\mathcal{R}\cap\mathsf{Tot})$ in $\mathscr{C}$.*

Proof. Let $(u,v) : l \longrightarrow r$ be an oplax square formed by total maps and such that $l \in \mathcal{L}$ and $r \in \mathcal{R}$. Then there exists a partial map δ that is an oplax diagonal morphism lifting l against r as in the following diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "l"'] \arrow[dr, phantom, "\leq", near start] & C \arrow[d, "r"] \arrow[dl, phantom, "\leq"', near end] \\
B \arrow[ur, "\delta"] \arrow[r, "v"'] & D
\end{tikzcd} \tag{5.xviii}$$

In particular the 2-cell $v \leq r\cdot\delta$ yields that $r\cdot\delta$ is total by Proposition 3.1 and thus, by Proposition 3.9, δ is total as well. Therefore δ is an oplax diagonal morphism in $\mathscr{C}$ lifting l against r. □

From here on we will denote $\mathcal{L}\cap\mathsf{Tot}$ by $\mathcal{L}_{\mathsf{Tot}}$ and $\mathcal{R}\cap\mathsf{Tot}$ by $\mathcal{R}_{\mathsf{Tot}}$.

Remark 5.13. *In the previous lemma, if we consider the* **Ord***-enrichment $\leq$, we have that $v \leq r\cdot\delta$ is an equality, since v is a total map. Similarly the upper triangle must be commutative as well and therefore the oplax weak orthogonality relation restricts to a weak orthogonality relation for commutative squares among total maps.*

Proposition 5.14. *Let $(\mathcal{L},\mathcal{R})$ be an oplax weak factorisation system for a category of partial maps $\mathcal{P}(\mathscr{C})$. Then any total morphism admits an $(\mathcal{L},\mathcal{R})$-factorisation consisting of total morphisms.*

Proof. Let us consider a partial map f and an $(\mathcal{L},\mathcal{R})$-factorisation $r_f\cdot l_f = f$. Our goal is to construct another $(\mathcal{L},\mathcal{R})$-factorisation with a total right component. In fact, if f is total, then its $\mathcal{L}$-component has to be total by the composition rules.

If f is any partial map (not total), then we consider the following factorisation

$$
\begin{tikzcd}
D_f \arrow[d, tail, "\sigma_f"'] \arrow[dr, "\varphi_f"] & \\
A \arrow[r, "f"', "+" marking] & B
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
D_f \arrow[d, tail, "{\varphi_l}^*(\sigma_r)"] \arrow[dr, "\widetilde{\varphi_l}"] \arrow[drr, bend left, "\varphi_f"] \arrow[dd, tail, bend right=60, "\sigma_f"'] & & \\
D_l \arrow[d, tail, "\sigma_l"'] \arrow[dr, "\varphi_l"] & D_r \arrow[d, tail, "\sigma_r"] \arrow[dr, "\varphi_r"] & \\
A \arrow[r, "l_f"', "+" marking] & M_f \arrow[r, "r_f"', "+" marking] & B.
\end{tikzcd}
$$

We first remark the following helpful fact. We consider the adjoint morphisms $\mu = (\mathrm{id}_{D_r}, \sigma_r) \dashv (\sigma_r, \mathrm{id}_{D_r}) = \nu$. We define $\overline{l}_f = \nu \cdot l_f$ and $\overline{r}_f = r_f \cdot \mu$ and by the counit 2-cells we obtain that

$$
\begin{cases}
\mu \cdot \overline{l}_f = \mu \cdot \nu \cdot l_f \leq l_f \\
\overline{r}_f \cdot \nu = r_f \cdot \mu \cdot \nu \leq r_f.
\end{cases}
$$

By directly computing we get that $r_f = \overline{r}_f \cdot \nu$. Furthermore, we have that $D_l = D_f$ if and only if $l_f = \mu \cdot \overline{l}_f$. This is easy to prove by explicitly computing the composition $\mu \cdot \nu \cdot l_f$ and remarking that both conditions are equivalent to have that ${\varphi_l}^*(\sigma_r)$ is an isomorphism.

Then we obtain the following factorisation

$$
\begin{tikzcd}
D_f \arrow[d, tail, "\sigma_f"'] \arrow[dr, "\varphi_f"] & \\
A \arrow[r, "f"', "+" marking] & B
\end{tikzcd}
\quad \longmapsto \quad
\begin{tikzcd}
D_f \arrow[d, tail, "\mathrm{id}_{D_f}"] \arrow[dr, "\overline{\varphi_l}"] \arrow[drr, bend left, "\varphi_f"] \arrow[dd, tail, bend right=60, "\sigma_f"'] & & \\
D_f \arrow[d, tail, "\sigma_f"'] \arrow[dr, "\overline{\varphi_l}"] & D_r \arrow[d, tail, "\mathrm{id}_{D_r}"] \arrow[dr, "\varphi_r"] & \\
A \arrow[r, "\overline{l}_f"', "+" marking] & D_r \arrow[r, "\overline{r}_f"', "+" marking] & B.
\end{tikzcd}
\tag{5.xix}
$$

Let us consider now a total morphism f. We proceed to study whether $\overline{l}_f \in \mathcal{L}$ and $\overline{r}_f \in \mathcal{R}$. Let us consider $\overline{r}_f$ and an oplax square $(u,v) : l \longrightarrow \overline{r}_f$, with $l \in \mathcal{L}$. We can write the following diagram

$$
\begin{tikzcd}
C \arrow[rr, "u"] \arrow[dr, "\mu \cdot u"] \arrow[dd, "l"'] & & D_r \arrow[dl, "\mu"'] \arrow[dd, "\overline{r}_f"] \\
\arrow[r, phantom, "\leq"] & M_f \arrow[dr, "r_f"] \arrow[d, phantom, "\leq"] & \\
D \arrow[ur, dashed, "\delta"] \arrow[rr, "v"'] & {} & B
\end{tikzcd}
$$

where the upper and right triangles are commutative and δ is an oplax diagonal morphism lifting $l \in \mathcal{L}$ against $r_f \in \mathcal{R}$. We consider a diagonal $\nu \cdot \delta$. Then considering the outer square we have

$$\begin{cases} \nu \cdot \delta \cdot l \leq \nu \cdot \mu \cdot u = u; \\ v \leq r_f \cdot \delta = \bar{r}_f \cdot \nu \cdot \delta. \end{cases}$$

Hence $\nu \cdot \delta$ is an oplax diagonal morphism for l against $\bar{r}_f$ and in conclusion $\bar{r}_f \in \mathcal{R}$.

Let us consider $\bar{l}_f$. For any $r \in \mathcal{R}$ and any oplax square $(u,v) : \bar{l}_f \longrightarrow r$, then we have the following diagram

$$\begin{tikzcd}
A \arrow[rr, "u"] \arrow[dr, "l_f"'] \arrow[dd, "\bar{l}_f"'] \arrow[r, phantom, shift right=4ex, "\leq"] & & C \arrow[dd, "r"] \\
& M_f \arrow[ur, dashed, "\delta'"] \arrow[dl, "\nu"'] \arrow[dr, "v\cdot\nu"] \arrow[ur, phantom, shift right=4ex, "\leq"] & \\
D_r \arrow[rr, "v"'] & & D,
\end{tikzcd}$$

and again the lower and the left triangles are commutative and δ' is an oplax diagonal morphism lifting $l_f \in \mathcal{L}$ against $r \in \mathcal{R}$. Now the diagonal morphism for the outer diagram is $\delta' \cdot \mu$ and the proof proceeds analogously by recalling that if f is total, then $D_l = D_f$. Indeed, we have the following 2-cells

$$\begin{cases} \delta' \cdot \mu \cdot \bar{l}_f = \delta' \cdot l_f \leq u; \\ v = v \cdot \nu \cdot \mu \leq r \cdot \delta' \cdot \mu. \end{cases}$$

Therefore we have that $\bar{l}_f \in \mathcal{L}$. $\square$

Remark 5.15. *We point out that the process presented above is not successful in general for the lax case. In fact, considering diagram* (5.xviii)*, it yields $u \leq \delta \cdot l$ and $r \cdot \delta \leq v$, which do not imply in general that δ is total. Moreover, we cannot deduce the commutativity of any triangle in the diagram, when the* **Ord***-enrichment is $\leq$. Nonetheless one can reproduce the same process of extracting a total factorisation from any lax factorisation of total maps. This process in the lax case is in fact successful, under the necessary condition that $D_l = D_f$, which is trivially satisfied by total maps.*

We conclude this section by proving the following proposition.

Proposition 5.16. *Let $\mathcal{P}_{\mathcal{S}}(\mathscr{C})$ be a category of partial maps and $(\mathcal{L},\mathcal{R})$ be an* $\mathrm{WFS}_{\mathrm{oplax}}$*. Then $(\mathcal{L}_{\mathsf{Tot}},\mathcal{R}_{\mathsf{Tot}})$ is an $\mathcal{S}$-stable* $\mathrm{WFS}_{\mathrm{oplax}}$ *for $\mathscr{C}$.*

Proof. We have proved before that the oplax weak orthogonality relation restricts to an oplax weak orthogonality relation among total maps, hence $\mathcal{L}_{\mathsf{Tot}} \pitchfork \mathcal{R}_{\mathsf{Tot}}$, and any morphism f has a $(\mathcal{L}_{\mathsf{Tot}},\mathcal{R}_{\mathsf{Tot}})$-factorisation as shown in (5.xix). We denote such factorisation by $f = r_f \cdot l_f$. Let f be a map in $\mathscr{C}$ such that $f \in {}^{\pitchfork}\mathcal{R}_{\mathsf{Tot}}$. The commutative square $\left(l_f, \mathrm{id}_B\right) : f \longrightarrow r_f$ admits an oplax diagonal morphism ρ_f, which must be total as well, and it yields the following 2-cells

$$\begin{cases} \rho_f \cdot f \leq l_f; \\ \mathrm{id}_B \leq r_f \cdot \rho_f. \end{cases}$$

We consider $g \in \mathcal{R}$ and an oplax square $(u,v) : f \longrightarrow g$ in $\mathcal{P}_{\mathcal{S}}(\mathscr{C})^2_{\mathrm{oplax}}$. Then we have the following diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "l_f"'] \arrow[dr, phantom, "\leq"] & C \arrow[dd, "g"] \\
D_f \arrow[ur, dotted, "\delta"] \arrow[d, "r_f"] \arrow[r, phantom, "\leq"] & {} \\
B \arrow[r, "v"'] \arrow[u, bend left, "\rho_f"] & D
\end{tikzcd}$$

where δ is the oplax diagonal morphism that lifts l_f against g for the oplax square $(u, v \cdot r_f)$. Then we obtain

$$\begin{cases} \delta \cdot \rho_f \cdot f \leq \delta \cdot l_f \leq u; \\ v \leq v \cdot r_f \cdot \rho_f \leq g \cdot \delta \cdot \rho_f. \end{cases}$$

We conclude that $f \in \mathcal{L}_{\mathsf{Tot}}$. Hence ${}^{\pitchfork}\mathcal{R}_{\mathsf{Tot}} = \mathcal{L}_{\mathsf{Tot}}$. A dual argument yields that $\mathcal{L}_{\mathsf{Tot}}^{\pitchfork} = \mathcal{R}_{\mathsf{Tot}}$.

We conclude the proof by checking that $(\mathcal{L}_{\mathsf{Tot}}, \mathcal{R}_{\mathsf{Tot}})$ is $\mathcal{S}$-stable. We consider $\varphi_f \in \mathcal{L}_{\mathsf{Tot}}$ and $\sigma \in \mathcal{S}$. Then we recall that in $\mathcal{P}_{\mathcal{S}}(\mathscr{C})$, (σ, id) forms the adjunction $(\mathrm{id}, \sigma) \dashv (\sigma, \mathrm{id})$. Moreover, $\mathcal{L}$ is closed under composition with right adjoint morphisms. We remark that the pullback $\sigma^*(\varphi_l)$ is the function component of the composition $(\sigma, \mathrm{id}) \cdot (\mathrm{id}, \varphi_l)$, which belongs to $\mathcal{L}$. Hence $\sigma^*(\varphi_l) \in \mathcal{L}_{\mathsf{Tot}}$ by Proposition 5.11, which yields the thesis. □

5.2.4 The bijection

We conclude illustrating the existence of a bijection between these two classes of factorisation systems. Let $\mathscr{C}$ be an **Ord**-category and $\mathcal{P}_{\mathcal{S}}(\mathscr{C})$ the category of partial maps for a class of admissible subobjects $\mathcal{S}$ such that any morphism in $\mathcal{S}$ is full and upper-closed. We denote by $\mathcal{S}$-$\mathbf{WFS}_{\mathrm{oplax}}(\mathscr{C})$ the class of $\mathcal{S}$-stable WFSs$_{\mathrm{oplax}}$ on $\mathscr{C}$. Furthermore, we denote by $\mathbf{WFS}_{\mathrm{oplax}}(\mathcal{P}(\mathscr{C}))$ the class of WFSs$_{\mathrm{oplax}}$ on $\mathcal{P}(\mathscr{C})$. Then we introduce the following two maps

$$\begin{tikzcd}
\mathbf{WFS}_{\mathrm{oplax}}(\mathcal{P}(\mathscr{C})) \arrow[r, bend left, "\Phi"] & \mathcal{S}\text{-}\mathbf{WFS}_{\mathrm{oplax}}(\mathscr{C}) \arrow[l, bend left, "\Psi"]
\end{tikzcd}$$

defined as follows

$$\begin{aligned} \Phi(\mathcal{L}, \mathcal{R}) &= \left(\overline{\mathcal{L}}, \overline{\mathcal{R}}\right); \\ \Psi(\mathcal{E}, \mathcal{M}) &= (\mathcal{E}_{\mathsf{Tot}}, \mathcal{M}_{\mathsf{Tot}}). \end{aligned} \tag{5.xx}$$

The first is obtained as defined in (5.viii) and the second as presented in Proposition 5.16. We proceed to show that the two functions are inverse.

Proposition 5.17. *Let $\mathscr{C}$ be an* **Ord***-category and $\mathcal{P}_{\mathcal{S}}(\mathscr{C})$ the category of partial maps for a class of admissible subobjects $\mathcal{S}$ such that any morphism in $\mathcal{S}$ is full and upper-closed. Then there exists an isomorphism of partially ordered sets between $\mathbf{WFS}_{\text{oplax}}(\mathcal{P}(\mathscr{C}))$ and $\mathcal{S}$-$\mathbf{WFS}_{\text{oplax}}(\mathscr{C})$.*

Proof. First we consider $\Psi(\Phi(\mathcal{E},\mathcal{M})) = \left(\overline{\mathcal{E}}_{\mathsf{Tot}},\overline{\mathcal{M}}_{\mathsf{Tot}}\right)$. By construction we have that $e \in \mathcal{E}$ if and only if $(\mathrm{id},e) \in \overline{\mathcal{E}}$, which is equivalent to have that $e \in \overline{\mathcal{E}}_{\mathsf{Tot}}$. Hence $\mathcal{E} = \overline{\mathcal{E}}_{\mathsf{Tot}}$ and similarly $\mathcal{M} = \overline{\mathcal{M}}_{\mathsf{Tot}}$.

Then we consider $\Phi(\Psi(\mathcal{L},\mathcal{R})) = \left(\overline{\mathcal{L}_{\mathsf{Tot}}},\overline{\mathcal{R}_{\mathsf{Tot}}}\right)$. Let l be a morphism in $\mathcal{L}$. By Proposition 5.11, we have that $(\mathrm{id},\varphi_l) \in \mathcal{L}$, hence $\varphi_l \in \mathcal{L}_{\mathsf{Tot}}$ and therefore $l \in \overline{\mathcal{L}_{\mathsf{Tot}}}$ by construction. On the other hand, if $l \in \overline{\mathcal{L}_{\mathsf{Tot}}}$, then $\varphi_l \in \mathcal{L}_{\mathsf{Tot}}$ and hence $(\mathrm{id},\varphi_l) \in \mathcal{L}$. We consider the partial map (σ_l,id), which is a right adjoint morphism in $\mathcal{P}_{\mathcal{S}}(\mathscr{C})$. Then we have trivially that $l = (\mathrm{id},\varphi_l)\cdot(\sigma_l,\mathrm{id})$ belongs to $\overline{\mathcal{L}_{\mathsf{Tot}}}$, since it is closed under composition with right adjoint morphisms. Hence $\mathcal{L} = \overline{\mathcal{L}_{\mathsf{Tot}}}$, and by similar arguments, $\mathcal{R} = \overline{\mathcal{R}_{\mathsf{Tot}}}$. This yields that Φ and Ψ are inverse functions. It is trivial to observe that inclusions are preserved by the Φ and Ψ. $\square$

More explicitly this classifies every $\text{WFS}_{\text{oplax}}$ on $\mathcal{P}_{\mathcal{S}}(\mathscr{C})$ as a factorisation system of the form $\left(\overline{\mathcal{L}},\overline{\mathcal{R}}\right)$ for some $\mathcal{S}$-stable $\text{WFS}_{\text{oplax}}$ $(\mathcal{L},\mathcal{R})$ in $\mathscr{C}$.

5.3 Factorisations for pointed categories of partial maps

Our goal for the following section is to discuss a process that gives rise to lax and oplax weak factorisation systems for certain pointed categories. The idea is to consider an **Ord**-enriched category such that every Hom-set has a least element morphism and such morphisms are *absorbent*, in the sense that whenever these morphisms are composed with any other morphism, the composition ends up being a least element map for the appropriate Hom-set. Indeed, one example of such maps is the class of zero-morphisms in a pointed category.

First we will discuss the general conditions for this construction, then we will apply this construction to categories of partial maps. We conclude the section showing that this construction is successful and has a complete description for $\mathcal{P}(\mathbf{Set})$.

5.3.1 Pointed categories

The setting for this subsection are pointed **Ord**-categories, i.e. **Ord**-categories with a zero-object which is both initial and final, and such that zero-maps are least elements with respect to their Hom-sets.

We consider the following class

$$\mathcal{O} = \{0_{A,B} \mid A,B \in \mathrm{Ob}(\mathscr{C})\}.$$

We recall that maps in $\mathcal{O}$ are *left absorbent* and *right absorbent*, in the sense that for every morphism $f : A \longrightarrow B$ in $\mathscr{C}$ we have that $f \cdot 0_{X,A} \in \mathcal{O}$ and $0_{B,Y} \cdot f \in \mathcal{O}$, for any $X,Y \in \mathrm{Ob}(\mathscr{C})$. Then we intend to study the lax and oplax weak orthogonal complements of $\mathcal{O}$.

First we consider $\mathcal{O}^{\pitchfork}$. Let $f : A \longrightarrow B$ be a morphism in $\mathcal{O}^{\pitchfork}$ and $(u,v) : 0_{X,Y} \longrightarrow f$. Then, by minimality of zero-maps, we have that the 2-cell $f \cdot u \le v \cdot 0_{X,Y}$ yields that $f \cdot u = 0_{X,B}$. From lax weak orthogonality follows that there exists a lax diagonal morphism δ in the diagram

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "0"'] \arrow[dr, phantom, "\leq", near start] & A \arrow[d, "f"] \\ Y \arrow[r, "v"'] \arrow[ur, "\delta"] \arrow[ur, phantom, "\leq", near end, shift right=3] & B. \end{tikzcd}$$

From the 2-cell $u \leq \delta \cdot 0_{X,Y} = 0_{X,A}$, follows again that $u = 0_{X,A}$. Hence $f \in \mathcal{O}^{\pitchfork}$ implies that for any u such that $f \cdot u = 0_{X,B}$, then $u = 0_{X,A}$, since the lax square $(u, \mathrm{id}_B) : 0_{X,B} \longrightarrow f$ admits a lax diagonal morphism.

On the other hand, if we consider f such that $f \cdot u = 0_{X,B}$ yields $u = 0_{X,A}$ for any morphism u, then we show that $f \in \mathcal{O}^{\pitchfork}$. Let us consider any lax square $(u, v) : 0_{X,Y} \longrightarrow f$. The equality $f \cdot u = 0_{X,B}$ implies that $u = 0_{X,A}$. We remark that $0_{Y,A}$ trivially constitute a lax diagonal morphism for the lax square.

We conclude that

$$\mathcal{O}^{\pitchfork} = \{ f \mid f \cdot u = 0 \ \Rightarrow \ u = 0 \} . \tag{5.xxi}$$

We consider ${}^{\pitchfork}\mathcal{O}$. Let $f : A \longrightarrow B$ be a morphism in ${}^{\pitchfork}\mathcal{O}$. Then, in particular, there exists a lax diagonal f^* for the lax square $(\mathrm{id}_A, \mathrm{id}_B) : f \longrightarrow 0$.

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\leq", near start] & A \arrow[d, "0_{A,B}"] \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, "f^*"] \arrow[ur, phantom, "\leq", near end, shift right=3] & B. \end{tikzcd}$$

We point out that the 2-cell $0_{A,B} \cdot f^* = 0_{B,B} \leq \mathrm{id}_B$ trivially exists by minimality of zero-maps. On the other hand, the 2-cell $\mathrm{id}_A \leq f^* \cdot f$ does not exist in general. We remark that the existence of a morphism f^* such that $\mathrm{id}_A \leq f^* \cdot f$, is also a sufficient condition for f to be in ${}^{\pitchfork}\mathcal{O}$. In fact, if we consider such a morphism f and a lax square $(u, v) : f \longrightarrow 0_{X,Y}$, then we have the following diagram

$$\begin{tikzcd} A \arrow[rr, "u"] \arrow[dd, "f"'] \arrow[dr, phantom, "\leq", near start] & & X \arrow[dd, "0_{X,Y}"] \\ & A \arrow[ur, "u"] & \\ B \arrow[ur, "f^*"] \arrow[rr, "v"'] & & Y \arrow[ul, phantom, "\leq", near start] \end{tikzcd}$$

and $u \leq u \cdot f^* \cdot f$ follows from the assumption on f, and $0_{X,Y} \cdot u \cdot f^* = 0_{B,Y} \leq v$ follows from minimality of zero-maps. In conclusion we have that

$$\mathcal{U} = {}^{\pitchfork}\mathcal{O} = \{ f \mid \mathrm{id}_A \leq f^* \cdot f \text{ for some } f \} . \tag{5.xxii}$$

Although we were not able to describe in general its complement we state a conjecture that points to the best candidate.

Conjecture 5.18. *The complement* $\mathcal{U}^{\pitchfork} = \{f | f \cdot f^* \leq \mathrm{id}_B \text{ for some } f^*\}$, *the intersection* $\mathcal{U}^{\pitchfork} \cap \mathcal{U}$ *being exactly the left adjoint morphisms.*

We briefly state the two counterparts for $\mathscr{C}^2_{\text{oplax}}$ that arise in a similar fashion.

1. Considering ${}^{\curlyvee}\mathcal{O}$ one can prove that

$$\mathcal{V} = \mathcal{O}^{\curlyvee} = \{f \mid \mathrm{id}_B \leq f \cdot f_* \text{ for some } f_*\} \tag{5.xxiii}$$

   and again we state the following conjecture.

Conjecture 5.19. *The complement* $\mathcal{V}^{\curlyvee} = \{f | f \cdot f_* \leq \mathrm{id}_A \text{ for some } f_*\}$, *the intersection being exactly the right adjoint morphisms.*

2. Then the left oplax complement is

$${}^{\curlyvee}\mathcal{O} = \{f \mid v \cdot f = 0_{A,Y} \ \Rightarrow \ v = 0_{B,Y}\}\,. \tag{5.xxiv}$$

Unfortunately, we were not able to describe in general the complements for these four instances of prefactorisation systems, but we point out that the four classes described are in general not trivial, therefore they generate a non-trivial lax or oplax weak prefactorisation system.

5.3.2 Factorisations for pointed categories of partial maps

In the following subsection, we apply the process described above on categories of partial maps. Along this section we will consider an **Ord**-category $\mathscr{C}$ with an initial object I and we assume that all initial morphisms i_X are admissible subobject morphisms, implying in particular that they are full and upper-closed. We notice that I is still an initial object in $\mathcal{P}(\mathscr{C})$ and that for any A, B the partial morphism $0_{A,B} = (i_A, i_B)$ is a least element in $\mathscr{C}(A,B)$. In fact for any partial map f, the arrow i_{D_f} shows that $0_{A,B} \leq f$. On the other hand if $f \leq 0_{A,B}$, then there exists an admissible subobject $s : D_f \longrightarrow I$, which yields that $f = 0_{A,B}$. We can consider now the class of minimal maps

$$\mathcal{O} = \{0_{A,B} = (i_A, i_B) : A \longrightarrow B \mid A; B \in \mathrm{Ob}(\mathscr{C})\}\,.$$

Remark 5.20. *We observe that in* $\mathcal{P}(\mathscr{C})$ *the initial object* I *of* $\mathscr{C}$ *is a zero-object whenever* I *is either a zero-object or a strict initial object. We recall that an initial object* I *is strict if every morphism* $f : A \longrightarrow I$ *is an isomorphism.*

This is true since the choice for the component φ_- becomes unique when the codomain is I under the said assumptions. We recall that cartesian closed categories, such as **Set**, **Cat**, any topos, and distributive categories, have strict initial objects.

Moreover, we remark that the property described in Remark 5.20 is not always needed. In fact, we are interested in the property of least elements being *left* or *right absorbent*, in the sense described above.

Lemma 5.21. *For any* $\mathcal{P}(\mathscr{C})$, *a minimal map* f *such that* $D_f = I$ *is right absorbent. Whenever* I *is actually a zero-object, then it is both left and right absorbent.*

This is trivial considering that the partial domain of the composition is a subobject of the partial domain of the first morphism and I admits only itself as a subobject. Therefore the hypothesis that I is a zero-object in $\mathcal{P}(\mathscr{C})$ is relevant only while discussing the left complements ${}^{\Uparrow}\mathcal{O}$ and ${}^{\Downarrow}\mathcal{O}$.

Hence the arguments of the previous paragraph can be adapted to $\mathcal{P}(\mathscr{C})$ under the assumptions above. We proceed making some remarks on the structures that arise in this context.

We consider ${}^{\Uparrow}\mathcal{O}$. By (5.xxii), we have that f belongs to ${}^{\Uparrow}\mathcal{O}$ if and only if there exists a partial map f^* such that $\mathrm{id}_A \leq f^* \cdot f$. We remark that if the **Ord**-enrichment on $\mathscr{C}$ is discrete, then this is equivalent to have $\mathrm{id}_A = f^* \cdot f$. We first remark that this yields that f is a total map. We write explicitly the equality

$$
\begin{tikzcd}
A \arrow[d, tail] \arrow[dr, "\widetilde{\varphi}_f"] \arrow[ddrr, bend left, "\mathrm{id}_A"] & & \\
A \arrow[d, tail, "\mathrm{id}_A"'] \arrow[dr, "\varphi_f"] & D_{f^*} \arrow[d, tail, "\sigma_{f^*}"'] \arrow[dr, "\varphi_{f^*}"] & \\
A \arrow[r, "f"', "\shortmid" marking] & B \arrow[r, "f^*"', "\shortmid" marking] & A,
\end{tikzcd}
$$

this shows that $\varphi_f = \sigma_{f^*} \cdot \widetilde{\varphi}_f$ is an admissible subobject morphism, since $\widetilde{\varphi}_f$ is a section and $\mathcal{S}$ contains all sections and is closed under composition. By Corollary 3.11, we conclude that f is a left adjoint morphism in $\mathcal{P}(\mathscr{C})$ and thus the lax prefactorisation system is the trivial $(\mathsf{LA}, \mathsf{All})$.

On the other hand, we look at $\mathcal{O}^{\Uparrow}$ as defined in (5.xxi). We remark that it is composed of those morphisms whose partial domain is maximal as a proper $\mathcal{S}$-subobject. Inspired by the example of partial maps, we chose to call such morphisms *dense domain partial maps* and we denote $\mathcal{O}^{\Uparrow}$ by $\mathcal{DD}$. In general we have that $\mathcal{DD} \supseteq \mathsf{Tot}$. In fact, if f is total and $f \cdot u = 0_{X,B}$, then $D_u = D_{f \cdot u} = I$, by the rules of composition. Therefore we have that $u = 0_{X,A}$. However, we observe that the other inclusion is not always true. In fact, we have the following counterexamples of partial morphisms which are not total, but have a dense domain:

- in Ab maps such as

$$
\begin{tikzcd}
\mathbb{Z} \arrow[d, tail, "2"'] \arrow[dr] & \\
\mathbb{Z} \arrow[r, "\shortmid" marking] & \mathbb{Z}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[d, tail, "i"'] \arrow[dr] & \\
\mathbb{Q} \arrow[r, "\shortmid" marking] & \mathbb{Z}
\end{tikzcd}
$$

  are not total, but it is easily proved that they have dense domains;

- for the category of topological spaces equipped with open maps, we have that a domain is dense exactly when the domain is a topologically dense subobject of the domain, so any morphism f such that $\sigma_f = j : [0,1[\longrightarrow [0,1]$ is not total and yet it has a dense domain.

Since total maps have always a dense domain and since $\left(\overline{\mathcal{S}}, \mathsf{Tot}\right)$ is a $\textsc{wfs}_{\mathrm{lax}}$, as shown in Section 5.1, we have that $\left(\overline{\mathcal{S}}, \mathsf{Tot}\right) \leq \left({}^{\Uparrow}\mathcal{DD}, \mathcal{DD}\right)$.

Analogously, we introduce the notation $\mathcal{DI}$ for the class ${}^{\pitchfork}\mathcal{O}$, and we call such maps *dense image maps*.

Finally we consider ${}^{\pitchfork}\mathcal{O}$ and the **Ord**-enrichment defined by the partial order $\leq$. If $f : A \longrightarrow B$ is a morphism in $\mathcal{O}^{\pitchfork}$, then there exists an oplax diagonal morphism δ for the oplax square $(\mathrm{id}_A, \mathrm{id}_B) : 0_{A,B} \longrightarrow f$. Thus we have the two 2-cells

$$\begin{cases} \delta \cdot 0_{A,B} \leq \mathrm{id}_A \\ \mathrm{id}_B \leq f \cdot \delta. \end{cases}$$

Since the identity is total, the second 2-cell is indeed an equality, therefore f is a split epimorphism. Conversely, if f has a right inverse f', then, in any oplax square $(u,v) : 0_{X,Y} \longrightarrow f$, the morphism $f' \cdot v$ yields

$$\begin{cases} f' \cdot v \cdot 0_{X,Y} = 0_{X,A} \leq u \\ f \cdot f' \cdot v = v. \end{cases}$$

Therefore $f' \cdot v$ is an oplax diagonal morphism. Hence $\mathcal{O}^{\pitchfork}$ is the class of left inverse morphisms LI.

We remark that, if f is a split epimorphism in the category of partial maps $\mathcal{P}(\mathscr{C})$, then φ_f is a split epimorphism in $\mathscr{C}$. This follows from the composition rules in $\mathcal{P}(\mathscr{C})$. Conversely, if f is such that φ_f admits a right inverse φ_f^*, then we easily see that the span $\left(\mathrm{id}_B, \sigma_f \cdot \varphi_f^*\right)$ is a right inverse of f. Hence we conclude that split epimorphisms in $\mathcal{P}(\mathscr{C})$ are

$$\overline{\mathsf{LI}} = \left\{ f \mid \varphi_f \text{ is a split epimorphism in } \mathscr{C} \right\}.$$

Even if it has been difficult to give a better description for such complements, we remark that in general these classes appear to be non-trivial.

5.3.3 Set with partial maps

We apply now the arguments discussed above to the category $\mathcal{P}(\mathbf{Set})$ and we show that the structures that arise from the previous construction have a complete description in this particular setting.

We recall that in this context the empty set $\varnothing$ is a zero-object, since it is a strict initial object in **Set**. In fact, for every object A, we have the unique initial and terminal morphisms defined as $\iota_A = (\mathrm{id}_\varnothing, i_A)$ and $\tau_A = (i_A, \mathrm{id}_\varnothing)$, where i_A is the initial morphism associated to A in **Set**. Furthermore, zero-maps are defined as

$$\begin{tikzcd} \varnothing \arrow[d, tail, "i_A"'] \arrow[dr, "i_B"] & \\ A \arrow[r, "\varnothing_{A,B}"', "\shortmid" marking] & B. \end{tikzcd}$$

We consider the class of morphisms

$$\mathcal{O} = \{ \varnothing_{A,B} \mid A, B \in \mathbf{Set} \}.$$

Then we study the complements of $\mathcal{O}$.

- We consider $\mathcal{DD}$. We have shown that $\mathsf{Tot} \subseteq \mathcal{DD}$ in the previous subsection. Conversely, in $\mathcal{P}(\mathbf{Set})$ if $f : A \longrightarrow B$ is not total, then there exists an element $a \in A$ such that $a \notin D_f$. If we consider the morphism

$$\begin{aligned} u : X &\longrightarrow A \\ x &\longmapsto a, \end{aligned}$$

then $f \cdot u = \emptyset_{X,B}$, but u is not a zero-map. Thus $\mathcal{DD} = \mathsf{Tot}$. Since lax weak orthogonal complement are unique and we have shown that ${}^{\pitchfork}\mathsf{Tot} = \overline{\mathcal{S}}$ in Section 5.1. then we conclude that $\left({}^{\pitchfork}\mathcal{DD}, \mathcal{DD}\right) = \left(\overline{\mathcal{S}}, \mathsf{Tot}\right)$.

- Regarding $\mathcal{U}$, since the **Ord**-enrichment considered on **Set** is discrete we have that $\left(\mathcal{U}, \mathcal{U}^{\pitchfork}\right) = (\mathsf{LA}, \mathsf{All})$, as discussed in the previous subsection.

- We turn our attention to $\mathcal{DI}$. We remark that, if f is a surjective map in $\mathcal{P}(\mathbf{Set})$, then it is a dense image map. Indeed, we point out that, for any $v : B \longrightarrow Y$, the partial domain $D_{v \cdot f}$ is the preimage $\varphi_f^{-1}(D_v)$. If $v \cdot f = \emptyset_{A,Y}$, then $\varphi_f^{-1}(D_v) = \emptyset$, but since φ_f is surjective, this implies that $D_v = \emptyset$. On the other hand, given a map f which is not surjective, then we have that the composition

$$\begin{tikzcd} D_f \arrow[d, tail, "\sigma_f"'] \arrow[dr, "\varphi_f"] & B \setminus Im(F) \arrow[d, tail, "i"'] \arrow[dr, "i"] & \\ A \arrow[r, "|" marking, "f"'] & B \arrow[r, "|" marking, "v"'] & B, \end{tikzcd}$$

is the zero-map $\emptyset_{A,B}$, but v is not a zero-map. We conclude that $\mathcal{DI} = \overline{\mathsf{Epi}}$. However we know by Example 5.5 that the oplax weak orthogonal complement of $\overline{\mathsf{Epi}}$ is $\overline{\mathsf{Mono}}$, thus $\left(\mathcal{DI}, \mathcal{DI}^{\curlyvee}\right) = \left(\overline{\mathsf{Epi}}, \overline{\mathsf{Mono}}\right)$.

- Finally we consider $\mathcal{V}$, which is the class of split epimorphisms under this **Ord**-enrichment, as proved in the previous subsection. In particular, since in **Set** split epimorphisms are exactly surjective maps, we conclude that $\mathcal{V} = \overline{\mathsf{Epi}}$. Again by Example 5.5, we conclude that $\left({}^{\curlyvee}\mathcal{V}, \mathcal{V}\right) = \left(\overline{\mathsf{Mono}}, \overline{\mathsf{Epi}}\right)$.

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